USAAAO 2024 - First Round

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1. David is walking down MIT's infinite corridor (latitude $42°21'33''$) when he suddenly sees the sun aligning with the window at the end of the corridor. Being the observational master he is, David immediately pulls out his compass and measures the Sun to be at an azimuth of $245.81°$. Forgetting to bring his jacket, he is painfully reminded as he walks outside that it has been less than 6 months since the previous winter solstice. Which of the following choices is closest to the current date? Assume the corridor is parallel to the surface of the Earth.

   (a) January 15

   (b) January 30

   (c) February 15

   (d) March 20

   (e) April 1

Solution:

Since the corridor is parallel to the surface of the Earth, we know that the Sun's altitude is 0 degrees. Define points on the celestial sphere as follows: A is David's current location, B is the North Celestial Pole, and C is the location of the Sun.

By the spherical law of cosines,

$$\cos a = \cos A \sin b \sin c + \cos b \cos c$$

By definition, A is the azimuth angle of $245.81°$, $a = 90° - \delta$ where δ is the declination of the Sun, and $c = 90° - l$ where $l = 42.36°$ is the latitude of Boston. Additionally, b is just 90 degrees minus the altitude of the Sun, so $b = 90°$.

Therefore, our equation simplifies to

$$\sin \delta = \cos A \cos l \implies \delta = \sin^{-1}(\cos A \cos l)$$

Calculating this gives $\delta = -17.63°$.

Now, we draw a second spherical triangle. Let A be the direction of the vernal equinox, B be the location of the North Celestial Pole, and C be the direction of the Sun.

Since the declination of the Sun is negative but it has been less than 6 months since the last winter solstice, we know that the point C is between the location of the winter solstice and point A, and so therefore angle $A = 90° + i = 113.44°$ where $i = 23.44°$ is the Earth's axial tilt. Also, we know $a = 90° - \delta$. And since A is on the equator, we must have that $c = 90°$. Using the law of cosines again,

$$\cos a = \cos A \sin b \sin c + \cos b \cos c$$

$$\sin \delta = -\sin i \sin b \implies b = \sin^{-1}\left(-\frac{\cos A \cos l}{\sin i}\right) = 47.64°$$

During 365.25 days, the Sun travels 360° across the celestial sphere, so it would take

$$\frac{47.64°}{360°} \cdot 365.25 \text{ days} = 48.34 \text{ days}$$

to reach the vernal equinox from its current position. Thus all we have to do is count backwards 48-49 days from March 20th to get an answer of January 30.

Answer: B

2. Abhay looks at the light curves for two main sequence stars A and B, which you can assume are blackbodies. A has its peak at a frequency two times as high as that of B. By looking at the depth of spectral lines, Abhay can also determine that A has higher metallicity than B. Abhay makes the following statements:

   P: A has higher absolute magnitude than B.

   Q: A is older than B.

   Which of the following is true?

   (a) P and Q are true.
   (b) P and Q are false.
   (c) P is true and Q is false.
   (d) P is false and Q is true.
   (e) We don't have sufficient information for one or more of these statements.

Solution:

A has a peak at a higher frequency/lower wavelength, thus it has a higher temperature by Wien's law. Thus, by Stefan Boltzmann's law we can say A has higher intensity. This will means a lower absolute magnitude.

A has higher metallicity, thus it is younger (newer stars are able to use metals created in supernovae, and thus have higher metallic content).

P and Q are both false.

Answer: B

3. Aliens in a nearby star system (located in a random direction from Earth) are looking for nearby planets using the transit method. What is the probability that they can see the Earth transit across the Sun?

   (Assume they observe our Sun over multiple years and their instruments are sensitive enough to detect any transit that occurs.)

(a) 100%
(b) 5.8%
(c) 0.93%
(d) 0.47%
(e) 0.15%

Solution:

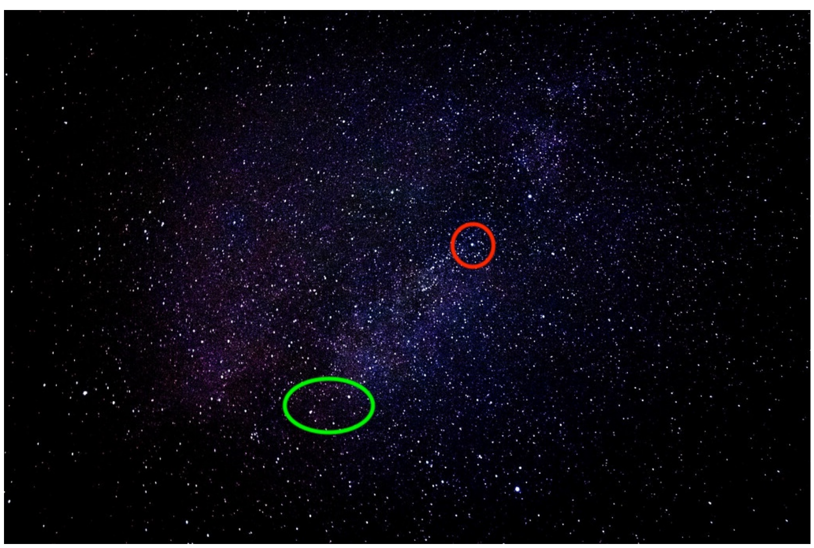
Using the small-angle approximation, a planet orbiting a distance a around a star of radius R can be seen to transit if the inclination of the orbital plane is less than R/a. We can think of this as tracing out a strip around the celestial sphere of width $2R/a$ (since the inclination can be in either direction), and the aliens will only see the transit if their star lies in this strip. The solid-angle area of this strip is $2\pi(2R/a)$, and the total solid angle of the celestial sphere is 4π, so the chance that a randomly selected star is within the strip is

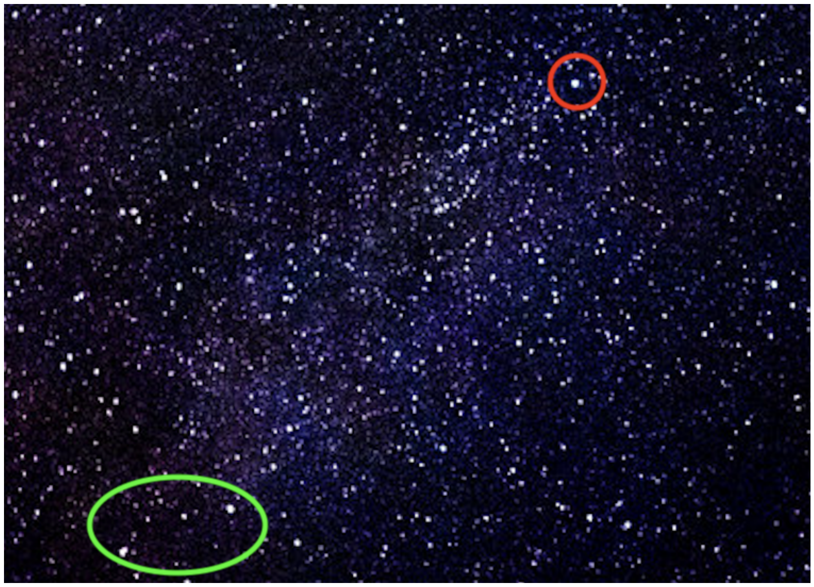
$$P(\text{transit}) = \frac{2\pi(2R/a)}{4\pi} = \frac{R}{a}$$

For Earth, this is about 0.47%.

$\boxed{\textbf{Answer: D}}$

4. Orion is observing the sky with two telescopes that he just made. Orion wrote down that the first telescope has a primary mirror with focal length $F_p = 2\text{m}$ and an eye piece with focal length $F_e = 30\text{mm}$. However, he does not know the specifications of his second telescope. Given that the full-field image on the left was taken by the first telescope, and the full-field image on the right was taken by the second telescope, which of the following choices could be the specifications of the second telescope?





(a) $F_p = 1\text{m}$ $F_e = 15\text{mm}$
(b) $F_p = 1\text{m}$ $F_e = 30\text{mm}$
(c) $F_p = 1\text{m}$ $F_e = 90\text{mm}$
(d) $F_p = 6\text{m}$ $F_e = 30\text{mm}$
(e) $F_p = 6\text{m}$ $F_e = 15\text{mm}$

Solution: Indicated by the red and green circles, the second image is roughly 3-4 times more magnified than the second picture. Magnification of a telescope equals $M = \frac{F_p}{F_e}$. Therefore, the only answer choices that allows the second telescope to produce a magnitude 3-4 times that of the first is answer choice D.

$\boxed{\textbf{Answer: D}}$

5. A supernova is triggered largely by neutrinos. In fact, 99% of the energy coming from the supernova is released in form of neutrinos. Over a time span of about three months, the supernova outputs visible light with power equivalent to 10 billion Suns. Assuming supernova neutrinos have mean energy of around 10 MeV, that all the power of the supernova is released during the time it is visible, and that all of the power released is released in the form of either visible light or neutrinos, estimate the number of neutrinos released.

(a) 10^{54}
(b) 10^{55}
(c) 10^{50}
(d) 10^{57}
(e) 10^{60}

Solution: Estimating the Luminosity of the supernova:

$$L_{SN} = 10^{10} \cdot 4 \cdot 10^{26} \text{ J s}^{-1} \cdot \frac{1 \text{ eV}}{1.609 \cdot 10^{-19} \text{ J}}$$

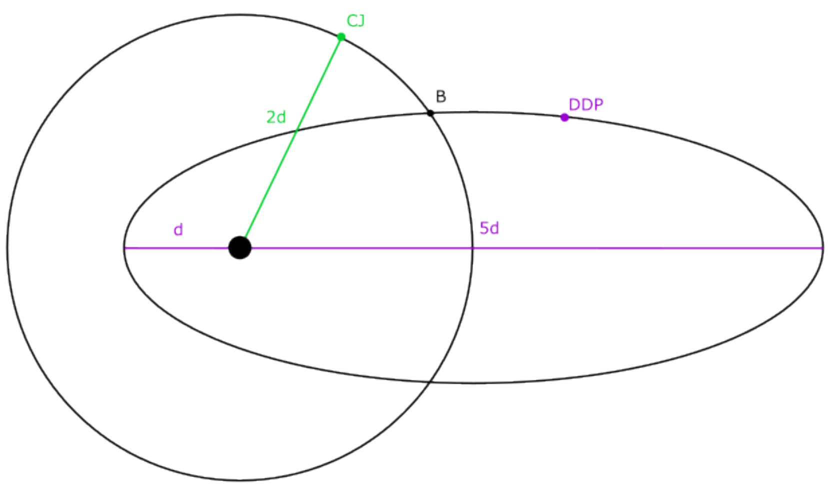
The luminosity in neutrinos is then:

$$L_\nu = 99 \cdot L_{SN}$$

Therefore, if we multiply this by the total amount of time and divide by the energy of the neutrino, we get the number of neutrinos:

$$N_\nu = \frac{L_\nu \cdot 90 \text{ d} \cdot \frac{86400\text{s}}{\text{d}}}{10 \cdot 10^6 \text{ eV}} \approx 10^{57}\text{neutrinos}$$

$\boxed{\textbf{Answer: D}}$



6. Newly discovered planets DDP and CJ are found to orbit a nearby star, as shown in the figure above. Planet CJ has a circular orbit with a radius of $2d$, while planet DDP moves in an elliptical orbit with an aphelion of d and a perihelion of $5d$. Their orbits intersect at location B in the figure. Additionally, through external analysis, planet DDP is found to be three times more massive than planet CJ. From the perspective of the star, what is the ratio of the angular momentum of planet DDP when it passes through point B to the angular momentum of planet CJ when it passes through point B? You may assume that the masses of both planets are significantly smaller than the mass of the star they orbit.

   (a) $\frac{3\sqrt{3}}{2}$

   (b) $\sqrt{\frac{15}{2}}$

   (c) $2\sqrt{3}$

(d) $\sqrt{30}$

(e) $\sqrt{\frac{10}{3}}$

Solution:

$\boxed{\textbf{Answer: B}}$

7. Arjun launches a 50 kg rocket with speed 10,405 m/s from the surface of the Earth and redirects it into a stable elliptical orbit. Upon analysis, he finds the area of the orbit to 1.438×10^{15} m^2. What is the approximate distance between the periapsis of the orbit and the center of the Earth? Assume no energy was lost in the redirection of the rocket into its new orbit.

(a) 10000 km

(b) 5000 km

(c) 2000 km

(d) 39000 km

(e) 15000 km

Solution:

$\boxed{\textbf{Answer: A}}$

8. The mass density of the Milky Way galaxy determines the orbital velocity of planets, stars, and other objects orbiting around its center. Assuming a constant surface mass density σ for the Milky Way and modeling it as a perfect circular disk, identify the dependence of the circular orbital velocity $v(r)$ of a point mass at radius r from the galaxy's center.

(a) $1/\sqrt{r}$

(b) $1/r$

(c) $\sqrt{r}$

(d) r

(e) $r^{3/2}$

Solution: From Newton's law, we have

$$\frac{G\sigma(\pi r^2)}{r^2} = \frac{v^2}{r} \tag{1}$$

which gives $v(r) \propto \sqrt{r}$.

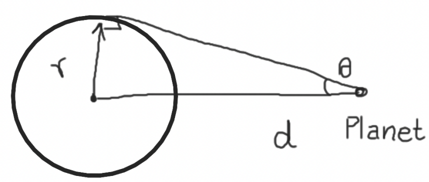
$\boxed{\textbf{Answer: C}}$

9. Let's assume that on an expedition mission to Mars, we take a telescope with 0.01 arcsecond angular resolution from earth. What is the ratio of the number of the stars we can measure the parallax distance to using this telescope on Mars compared to earth? The semimajor axis of Mars is 1.524 AU.

   (a) 0.25
   (b) 0.5
   (c) 1
   (d) 4
   (e) 8

Solution:

According to the figure below,



for a given small θ we can have: $\tan(\theta) \approx \theta \approx \frac{r}{d} \rightarrow \theta = \text{const.} = \frac{r_e}{d_e} = \frac{r_m}{d_m}$

$\frac{d_m}{d_e} = \frac{r_m}{r_e}$

Let N be the number of observed stars:

$N \propto d^3 \rightarrow \frac{N_m}{N_e} = (\frac{d_m}{d_e})^3 = (\frac{1.5}{1})^3 \approx 4.$

$\boxed{\textbf{Answer: D}}$

10. Ben Chen is an alien living on a system identical to earth, except his planet's obliquity is 0°. Located at latitude 42.20°, he wants to observe M52. Due to the open cluster being so dim, Ben needs perfect conditions to observe M52. Due to atmospheric effects, M52 can only be observed above an altitude of 30°. Additionally, it must be during astronomical twilight (when the Sun is more than 18° below the horizon). Of the following dates, which is the earliest after the vernal equinox that Ben can observe the cluster? The coordinates of M52 are approximately $\alpha = 0\text{h}$ and $\delta = 60°$.

    (a) April 21st
    (b) June 21st
    (c) August 21st
    (d) October 21st
    (e) December 21st

Solution:

Drawing a spherical triangle between the NCP, zenith, and M52 reveals that the hour angle at which M52 is at altitude 30° is $\pm$ 6h51m. Drawing a spherical triangle between the Sun, NCP, and zenith reveals that the Sun's hour angle at astronomical twilight is $\pm$ 7h39m. Since the Sun's rises later throughout the year, relative to M52, the first case where M52 is observable is when M52 reaches 30° right before the end of astronomical twilight. This means the Sun's right ascension is 7h39m - 6h51m = 48m, which occurs 13 days after March 21st.

Answer: A

11. Joe lives at the bottom of a vertical cylindrical hole with a radius of 10 m at a depth of 10 km below the surface. He sees the Sun directly through the opening of the hole for a couple days twice a year, around November 2nd and February 9th. Which of the following is Joe's latitude?

(a) $42^\circ 22'$N
(b) $19^\circ 27'$N
(c) $4^\circ 43'$N
(d) $14^\circ 38'$S
(e) $34^\circ 36'$S

Solution:

This question and the following one are inspired by Chapter 30 of *What If? 2* by Randall Munroe.

The declination of an object at zenith equals the latitude. The Sun's declination is negative in November and February, and the absolute value of the Sun's declination never exceeds the obliquity 23.44°, so the only possibility is $14^\circ 38'$S.

Answer: D

12. In the same scenario as the question above, what is the longest possible time interval that direct sunlight reaches anywhere in the bottom of the hole in a single day?

(a) 28 sec
(b) 2 min 8 sec
(c) 2 min 26 sec
(d) 2 min 35 sec
(e) 2 min 41 sec

Solution:

The portion of the sky visible anywhere at the bottom of the hole is a circle centered at the zenith, with angular radius

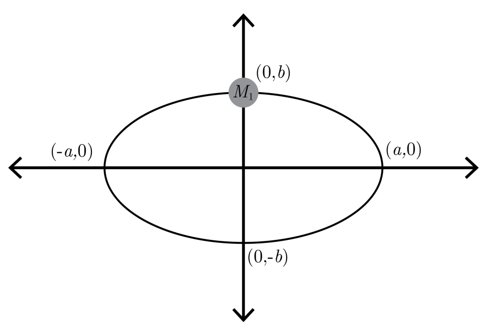
$$\rho = \tan^{-1}\left(\frac{10\,\mathrm{m}}{10\times 10^3\,\mathrm{m}}\right),$$

so there is direct sunlight at the bottom of the hole whenever zenith distance of the center of the Sun is less than this radius, plus the angular radius of the Sun $\theta_\odot$. If the center of the Sun passes directly overhead, the Sun travels a total of $2(\rho+\theta_\odot)$ across the sky while it is visible. The declination of the Sun is $\delta = -14°38'$ by the previous question, so the total angular distance covered by the Sun over the course of a 24 hr day is $360° \cos\delta$ (the Sun does not traverse a great circle when it is not on the equator). Therefore, the total time is

$$\begin{aligned}(24\,\mathrm{hr})\cdot\frac{2(\rho+\theta_\odot)}{360°\cos\delta} &= (24\cdot 60\,\mathrm{min})\cdot\frac{2\left(\tan^{-1}(10/10^4)+\tan^{-1}(6.96\times 10^8/1.496\times 10^{11})\right)}{360°\cos\left(-14°38'\right)}\\ &= 2.68\,\mathrm{min}.\end{aligned}$$

Answer: E

13. Samvit observes a binary star system of masses M_1 and M_2. Unfortunately, the star with mass M_2 is too dim for him to observe it, leading to the following snapshot below.



What could Samvit hypothesize to be the position and mass of the other star at this instant that would be consistent with the laws of physics and the orbit snapshot that he sees? To be clear, he has *no knowledge* of the value of M_2 or the period of the binary system.

(a) At $(\sqrt{a^2-b^2}, 0)$ with mass $M_1 + M_2$

(b) At $(0, -b)$ with mass M_1

(c) At $(\sqrt{a^2-b^2}, 0)$ with mass $\frac{M_1 M_2}{M_1+M_2}$

(d) At $(-2\sqrt{a^2-b^2}, -b)$ with mass M_1

(e) At $(\frac{M_2}{M_1}\sqrt{a^2-b^2}, 0)$ with mass M_2

Solution: The trick to this question is that the center of mass of the system has to lie at the focus of the ellipse, and this is the only thing that matters if you have no knowledge of the other parameters in the system. In fact, if you had the period of the binary system, you could fully determine the value of M_2 in terms of M_1 and the other parameters of the problem. So, we can simply go through the choices and see which choice places the center of mass of the system at one of the foci, $(\pm\sqrt{a^2 - b^2}, 0)$, and choice **(d)** does.

Answer: D

14. The surface of the Sun exhibits differential rotation, with different rotational periods at different latitudes. We can measure this rotation speed using Doppler spectroscopy or by tracking the motion of sunspots.

    If the rotation speed of the Sun's surface at the equator is 2021 m/s, and at 60° South is 809 m/s, how long would it take for a sunspot at the equator to do a full extra lap around the Sun compared to a sunspot at 60° South?

    (a) 6.2 days
    (b) 25.0 days
    (c) 31.2 days
    (d) 41.7 days
    (e) 126 days

Solution:

A point at latitude l on the surface of a body of radius R is located a distance $R\cos l$ away from the axis of rotation. Thus, the total distance d a point on the surface needs to travel for one full rotation is $d = 2\pi R\cos l$. We can use this to find the orbital period as a function of the rotational velocity:

$$T = \frac{d}{v} = \frac{2\pi R\cos l}{v}$$

Plugging in the given values as well as the Sun's radius for R, we get $T = 25.0$ days at the equator, and $T = 31.2$ days at 60° latitude.

The time it takes for one sunspot to do a full extra lap around the Sun relative to another is conceptually identical to the synodic period between two planets, and can be calculated by the same formula:

$$\frac{1}{T_{\text{lap}}} = \frac{1}{T_1} - \frac{1}{T_2}$$

$$T_{\text{lap}} = 126$$

days

Answer: E

15. Two protons A and B lie in the solar interior. In the rest frame of proton A, the proton B approaches it radially from a large distance with speed $0.9c$. In the rest frame of proton A,

identify the radius of the "classically forbidden" region for proton B (i.e. the region in which proton B cannot enter).

(a) 6.6×10^{-12} m
(b) 4.1×10^{-15} m
(c) 2.3×10^{-15} m
(d) 3.8×10^{-18} m
(e) 1.2×10^{-18} m

Solution:

Let r_c denote the radius of the classically forbidden region for proton B in the rest frame of proton A. This is also the radial position of the classical turning point of motion for proton B. Applying conservation of (relativistic) energy, we find

$$(\gamma(v) - 1)m_p c^2 = \frac{e^2}{4\pi\epsilon_0 r_c}, \tag{2}$$

where $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$ denotes the Lorentz factor.

$\boxed{\textbf{Answer: E}}$

16. Consider a sun-planet-moon system. The rotation period of the planet is 2 days. The period of revolution of the moon around the planet is 42 days while that of the planet around the sun is 420 days. What is the length of the lunar cycle as seen from the planet? You can assume the the direction of planetary rotation, planetary revolution and lunar revolution is the same.

(a) 42.1 days
(b) 44.3 days
(c) 46.7 days
(d) 50.5 days
(e) 53.1 days

Solution:

Synodic period i.e. length of lunar cycle is given as $\frac{1}{T_s} = \frac{1}{42} - \frac{1}{420}$.

$\boxed{\textbf{Answer: C}}$

17. The Extremely Large Telescope (ELT) is an optical telescope under construction in Chile. The primary mirror has been planned to have a diameter of 39.3 m making it largest optical telescope ever built. One of the goals for this telescope is the direct imaging of exoplanets. Consider an exoplanet at a distance of 1 A.U. from a star. What is the maximum distance from Earth of such a star-exoplanet system in which the ELT can resolve the exoplanet separately from the star? Ignore atmospheric seeing and assume optical wavelength to be 500 nm.

(a) 112 pc

(b) 212 pc
(c) 312 pc
(d) 412 pc
(e) 512 pc

Solution: Use the Rayleigh criterion to calculate the best-case resolution

$$\theta_m = \frac{1.22\lambda}{D}$$

where D is the diameter of the primary mirror.

The maximum distance to the stellar system d is calculated as $d\theta_m = 1 A.U.$. Converting A.U. to parsecs, we get $d = 312$ pc.

Answer: C

18. At 6am on March 20th, as the Sun is rising, Leo, who is at $(40°\text{N}, 75°\text{W})$, plants a stick vertically on the ground. At that moment, he marks out a (finite) line on the ground in the direction of the shadow of the stick at that moment, labeling it with the current time. Every hour afterwards, on the hour, he marks out a new line in the current direction of the shadow, until the sun sets at 6pm.

Three months later, Leo returns to the same spot, where the vertical stick and lines remain. Again, every hour on the hour, he marks out a line in the current direction of the shadow, until the sun sets.

Let α_{12} and α_6 be the azimuths of the lines drawn in the spring at 12pm and 6pm, and β_{12} and β_6 be the azimuths of the lines drawn in the summer at 12pm and 6pm. Which of the following statements is true? Ignore atmospheric effects and the equation of time.

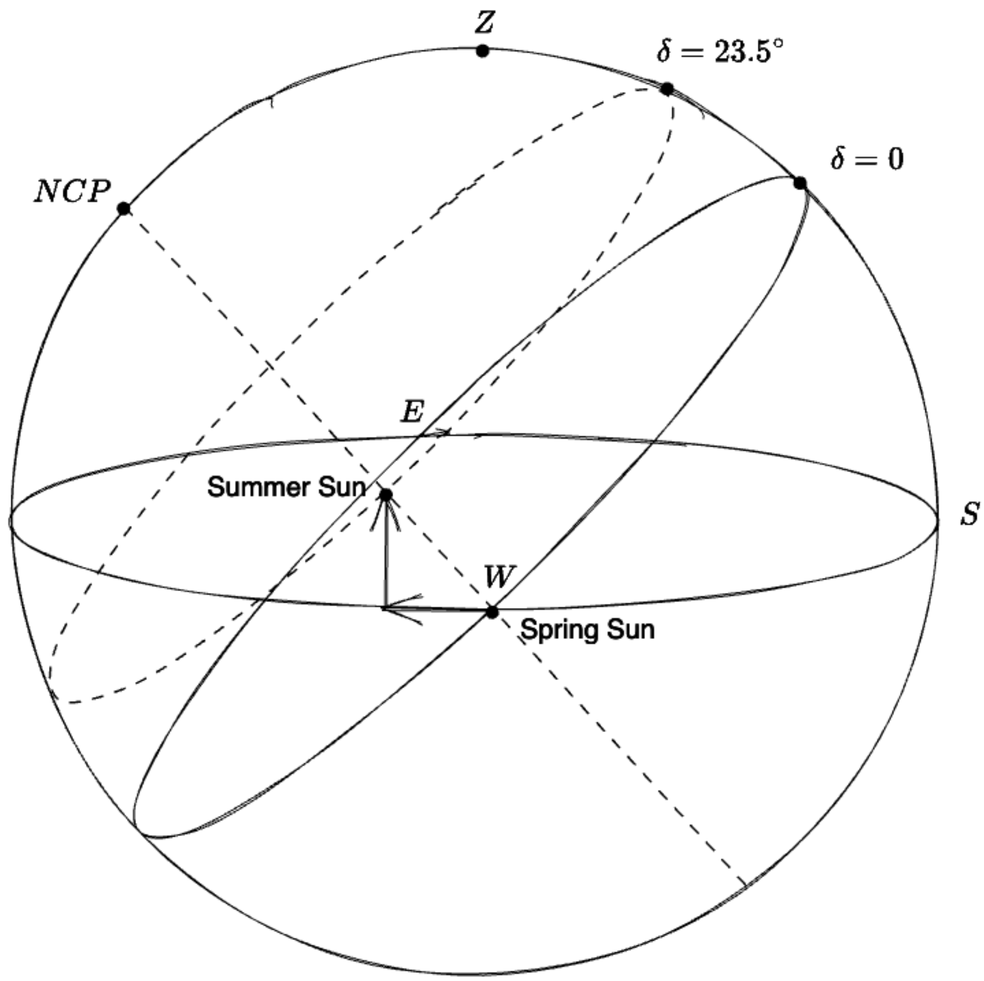
(a) $\alpha_{12} = \beta_{12}, \alpha_6 = \beta_6$
(b) $\alpha_{12} > \beta_{12}, \alpha_6 = \beta_6$
(c) $\alpha_{12} < \beta_{12}, \alpha_6 = \beta_6$
(d) $\alpha_{12} = \beta_{12}, \alpha_6 > \beta_6$
(e) $\alpha_{12} = \beta_{12}, \alpha_6 < \beta_6$

Solution:

As the stick is vertical and perpendicular to the ground, the line of the shadow is opposite the azimuth of the Sun, so angles between lines equal changes in azimuth of the Sun.

March 20th is the spring equinox, and solar time corresponds with local time (as the Sun rises at 6am and sets at 6pm, and 75°W is perfectly in the center of the UTC-5 time zone). Therefore, the Sun is on the meridian at 12pm and on the horizon due west at 6pm.

Three months later is the summer solstice. As solar time corresponds with local time and we ignore the equation of time, the Sun must still be on the meridian at noon, and at 6pm, the hour angle of the Sun must be 90°. However, as the Sun is traveling on a small circle with declination $\delta = 23.5°$, the Sun ends up North of West, with a greater azimuth, at this point (see diagram).



Therefore, $\alpha_{12} = \beta_{12}$ and $\alpha_6 < \beta_6$.

Answer: E

19. Leo then realizes that, in order for a single set of hour markings to accurately describe the time over the course of an entire year, the stick may need to be tilted away from the vertical position. More specifically, consider straight lines drawn on the ground from the base of the stick; the shadow at a certain fixed time of day, on different days of the year, should always lie on the same line. Measured as an angle from the vertical, how much does the stick need to be tilted, and in which direction?

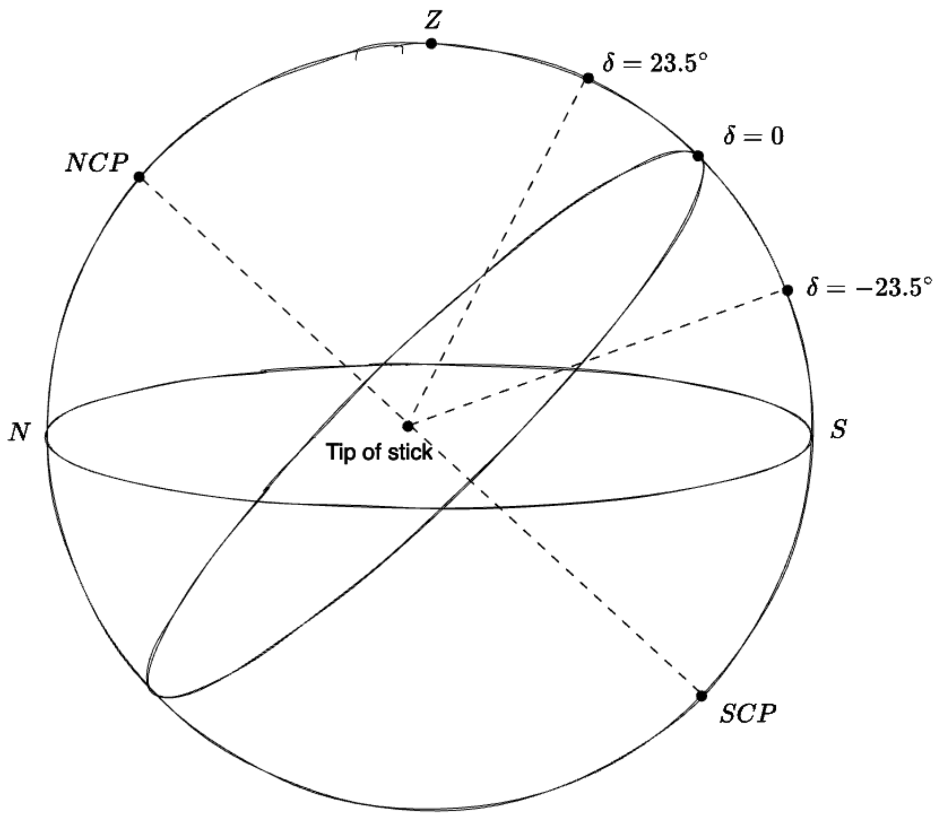
    (a) 0° (no tilt needed)
    (b) 40° towards the North
    (c) 50° towards the North
    (d) 40° towards the South

(e) $50°$ towards the South

Solution:

Place the tip of the stick at the center of the celestial sphere. Consider the position of the Sun at a certain fixed time of day, on different days of the year. For the same time of day, the hour angle stays the same, but the declination of the Sun varies between $\delta = -23.5°$ to $23.5°$, encompassing an arc of a great circle passing through the celestial poles.

Then, the shadow of the tip of the stick must always be in the plane containing the center and the arc. As the shadows on different days form a line, and the lines on the ground pass through the base of the stick, the base of the stick must also be in this plane.



Repeating this argument for different times of day, we get that the base of the stick must be in the intersection of planes defined by arcs at different hour angles. As all these planes pass through the North and South celestial poles, the intersection is the line joining the NCP and SCP, and the stick base must be on this line.

Therefore, the stick must point (upwards) in the direction of the NCP. As the altitude of the NCP is the latitude, the NCP is $40°$ above the horizon, or $50°$ tilted from the zenith towards the North.

Answer: C

20. Imagine a very long cylindrical planet that has a satellite orbiting around it. Considering that

the average density of the planet is ρ and the radius is R, find the expression that relates the period P of the satellite with its distance d to the center of the planet.

(a) $\frac{2d}{R}\sqrt{\frac{2\pi}{G\rho}}$

(b) $\frac{d}{R}\sqrt{\frac{2\pi}{G\rho}}$

(c) $\frac{d}{2R}\sqrt{\frac{4\pi}{G\rho}}$

(d) $\frac{d}{R}\sqrt{\frac{4\pi}{G\rho}}$

(e) $\frac{d}{R}\sqrt{\frac{\pi}{G\rho}}$

Solution:

Due to the symmetry, and using Gauss' Law for gravitation, we know that the gravity at a distance d from the center is given by $g \cdot S = -4\pi G M_{in}$. Our Gaussian shape will thus be a cylinder of radius d and length L. So, $S = 2\pi dL$, and the internal mass is given by $M_{in} = \rho\pi R^2 L$. Therefore, we have that:

$$g \cdot 2\pi dL = -4\pi G\rho\pi R^2 L \Rightarrow g(\vec{r}) = -\frac{2\pi G\rho R^2}{d}\hat{r}$$

Now, knowing that $F_g = F_{cp} \Rightarrow mg = m\omega^2 d = m(2\pi/P)^2 d$, we can find the period:

$$P = \frac{d}{R}\sqrt{\frac{2\pi}{G\rho}}$$

Answer: B

21. The problem of magnetic monopoles — that is, the apparent absence of magnetic monopoles in the universe — arises from the fact that some modern physical theories (such as string theory) predict that the number density of magnetic monopoles at the time of their creation was $n_M(t_{GUT}) \approx 10^{82}\,\mathrm{m}^{-3}$. The inflation theory provides a possible solution to this problem, as the exponential expansion of the primordial universe would "dilute" the monopoles. Calculate, approximately, how much the universe expanded during the inflationary period so that today the probability of a single magnetic monopole existing in the observational universe is 1%. Consider that the beginning of inflation coincides with the time of the creation of magnetic monopoles, and that the universe is flat (Euclidean geometry can be used on large scales). You can use that the diameter of the observational universe is 28.5 Gpc, and that between the end of inflation and today, the universe has linearly expanded by a factor of 5×10^{26}.

(a) e^{40}

(b) e^{50}

(c) e^{55}

(d) e^{65}

(e) e^{85}

Solution:

Let V_0 be the volume of the universe and R_0 its radius nowadays. The numerical density of magnetic monopoles today can be calculated by $n_M(t_0) = p_0/V_0 = 3p_0/(4\pi R_0^3)$, where p_0 is the probability of existing a monopole in the current universe. Besides that, we can relate $n_M(t_0)$ with $n_M(t_{GUT})$ by $n_M(t_0) = n_M(t_{GUT}) \cdot f_0^{-3} \cdot f_N^{-3}$, where f_0 is how much the universe expanded between the end of inflation and today, and f_N is how much it expanded during the inflationary period. Making the equality between these two relations, we have:

$$n_M(t_0) = n_M(t_{GUT}) \cdot f_0^{-3} \cdot f_N^{-3} = \frac{3p_0}{4\pi R_0^3} \Rightarrow f_N = n_M(t_{GUT}) f_0^{-3} \frac{4\pi R_0^3}{3p_0}$$

$$\Rightarrow f_N \approx e^{65}$$

Answer: D

22. Tara is investigating a new interesting type of stars she decides to call the X stars. She observes that in an X star of mass M and radius R, the gas pressure in the center of the star is proportional to $\frac{M^3}{R^5}$. What is the temperature at the center of the star proportional to?

    (a) MR
    (b) $\frac{M}{R}$
    (c) M^2R^2
    (d) $\frac{M^2}{R^2}$
    (e) $const$

Solution:

From the equation of the ideal gas we have:

$$P \propto \rho T$$

substituting we get:

$$\frac{M^3}{R^5} \propto \frac{M}{R^3} T \rightarrow T \propto \frac{M^2}{R^2}$$

Answer: D

23. VOIDED A recently discovered star *Recentus* in a nearby galaxy has been confirmed to have a luminosity of $4L_\odot$ with a radius of $4R_\odot$. What is the approximate frequency of peak emission of *Recentus*?

    (a) 2×10^{14} Hz
    (b) 4×10^{14} Hz
    (c) 6×10^{14} Hz
    (d) 8×10^{14} Hz

(e) 10×10^{14} Hz

Solution:

The luminosity L of a star in terms of its radius R and temperature T can be written using Stefan-Boltzmann's Law:

$$L = 4\pi R^2 \sigma T^4$$

$\frac{L}{(R^2T^4)}$ is constant, and we know that the luminosity and radius of *Recentus* are $L_R = 4L_\odot$ and $R_R = 4R_\odot$:

$$\frac{L_R}{(R_R^2 T_R^4)} = \frac{L_\odot}{(R_\odot^2 T_\odot^4)}$$

$$T_R = \frac{T_\odot}{\sqrt{2}}$$

Using Wein's law, the peak emission wavelength of this star is $\lambda_R = \lambda_\odot\sqrt{2} \approx 710$ nm and the corresponding peak emission frequency is 4.2×10^{14} Hz. Note that $\lambda_\odot = 502$ nm can be calculated from Wein's constant and $T_\odot$ which are in the TOC.

Answer: B

24. Tara has become obsessed with X stars and decides to look for them in other galaxies. She observes one on the edge of a galaxy with radius $r = 15 \times 10^3 pc$. She estimates that the star is moving with at a speed of $v = 270 km/s$. What is a good estimate for the mass of the galaxy in unit mass of the sun?

(a) 1×10^6
(b) 1×10^{11}
(c) 1×10^{18}
(d) 1×10^{28}
(e) 1×10^{41}

Solution: For a galaxy with mass M and an X star with mass m

$$\frac{GMm}{R^2} = \frac{mv^2}{R}$$

$$M = \frac{v^2 R}{G} =$$

$r = 15 \times 10^3 pc$ and $v = 270 km/s$, so $M = 2.5 \times 10^{11} M_\odot$

This treats the galaxy as a rigid body of mass M and that the X star is at the edge of the galaxy so the rest of the mass is concentrated inside the galaxy. The closest answer is $\approx 1 \times 10^{11} M_\odot$

$\boxed{\textbf{Answer: B}}$

25. There is a galaxy at redshift 0.5 for which we have a measurement for apparent bolometric magnitude to be 22. With a standard candle in that galaxy, we have found its luminosity distance to Earth to be 2.8 Gpc. Estimate the luminosity of this galaxy.

    (a) $10^{10} L_\odot$
    (b) $10^{12} L_\odot$
    (c) $10^{11} L_\odot$
    (d) $10^{13} L_\odot$
    (e) $10^{15} L_\odot$

Solution:

From the luminosity distance D in parsecs and the apparent magnitude m, we can calculate the absolute magnitude M of the galaxy:

$$m = M - 5 + 5log_{10}D$$

The absolute magnitude of this galaxy is -20.24. From the absolute magnitude we can calculate the luminosity of the star in terms of zero-point luminosity $L_0 = 3.01 \times 10^{28}$ W as:

$$M = -2.5log_{10}\frac{L}{L_0}$$

The luminosity $L = 10^{\frac{-20.24}{-2.5}} L_0$. Hence, $L \approx 10^{10} L_\odot$.

$\boxed{\textbf{Answer: A}}$

26. An astronomer wants to design a telescope so that the full moon fills the entire FOV of the telescope. She uses an eyepiece with a FOV of 60°. If the focal length of the eyepiece is 25mm, what will the focal length of the chosen telescope be?

    Note that the angular diameter of the moon is 0.5°. Never look at the moon through a telescope without proper precautions!

    (a) 2000mm
    (b) 1500mm
    (c) 6000mm
    (d) 3000mm
    (e) 1000mm

Solution:

We can write magnification of a telescope in two ways:

$$m = \frac{f_{\text{telescope}}}{f_{\text{eyepiece}}} = \frac{\text{FOV}_{\text{eyepiece}}}{\text{FOV}_{\text{telescope}}} = \frac{60}{0.5}$$

$\boxed{\textbf{Answer: D}}$

27. The Cosmic Microwave Background is made of light that was replied when the Universe first became transparent. It is a blackbody spectrum with temperature equal to the current temperature of the Universe. We observe the peak wavelength of the CMB to be at 1.063 millimeters. When the CMB was released, we can theoretically predict the temperature of the universe to be 3000 Kelvins. How much larger was the density of matter when the CMB was released than now? Select the closest answer.

   (a) 10^3
   (b) 10^6
   (c) 10^9
   (d) 10^{12}
   (e) 10^{15}

Solution:

We use Wien's law to find that the temperature of the CMB is 2.726 Kelvin. Then note $T = \frac{T_0}{a}$ where a is the scale factor. We find $a = 1100.5$. Now density of matter changes as $\rho = \frac{\rho_0}{a^3}$. Thus the answer is $(1100.5)^3$ which is closest to 10^9

$\boxed{\textbf{Answer: C}}$

28. **Questions 28-30 build upon the same prompt. Use data from any of the questions for any of the other questions.**
   Moving into MIT for the start of the spring semester, Austin is flying from Lubbock, Texas $(33.58°\text{N}, 101.84°\text{W})$ to Boston, MA $(42.36°\text{N}, 71.06°\text{W})$. However, when he lands he finds that he is not in Boston. The pilots entered the latitude coordinate incorrectly! But Austin remembers that the plane left Lubbock at a bearing of $63°$. Assume that the flight still took the shortest path to the current destination. Where is Austin now?

   (a) $(51.46°\text{N}, 71.06°\text{W})$
   (b) $(48.46°\text{N}, 71.06°\text{W})$
   (c) $(45.46°\text{N}, 71.06°\text{W})$
   (d) $(41.46°\text{N}, 71.06°\text{W})$
   (e) $(39.46°\text{N}, 71.06°\text{W})$

Solution:

Consider the spherical triangle with vertices Lubbock, the new location, and the North Pole, and sides being the great circle paths connecting them (the plane travels in a great circle, since it is traveling on the shortest path). Let the side connecting Boston and the North Pole to be x. Side between Lubbock and North Pole = $(90^\circ - 33.58^\circ) = 56.42^\circ$, and angle at North Pole = $(101.84^\circ - 71.06^\circ) = 30.79^\circ$. We can solve for x using the four-parts formula:

$$\cos(56.42^\circ)\cos(30.79^\circ) = \sin(56.42^\circ)\cot(x) - \sin(30.79^\circ)\cot(63^\circ)$$

$$x = 48.54^\circ.$$

This means the latitude of the new location is $90^\circ - 48.54^\circ = 41.46^\circ$.

$\boxed{\textbf{Answer: D}}$

29. After that slight headache, Austin is back at MIT in Boston! For his astronomy research, he is observing the LARES satellite which is a ball of diameter 36.4 cm made out of THA-18N (a tungsten alloy). It orbits at a distance 1450 km from the surface of the Earth and at an inclination of 69.49° relative to the equatorial plane. What is the highest altitude Austin can point his telescope if he wants to observe LARES at its highest latitude?

(a) 9.4°

(b) 14.4°

(c) 18.4°

(d) 23.4°

(e) 33.4°

Solution:

Let the radius of the Earth be R, the distance of LARES from the surface of the Earth be H, LARES' inclination be θ, latitude of Boston be ϕ, and the distance of LARES from Boston be d. Consider the triangle with vertices of the center of the Earth, Boston, and LARES' maximal latitude position. (This triangle lies in the plane containing the North and South poles). Then from Law of Cosines

$$d^2 = (H+R)^2 + R^2 - 2R(H+R)\cos(\theta - \phi)$$

$$d = 3615 \text{ km}.$$

Let the corresponding altitude be α. Then from Law of Sines

$$\frac{\sin(90^\circ + \alpha)}{(H+R)} = \frac{\sin(\theta - \phi)}{d}$$

$$\alpha = 9.4^\circ$$

$\boxed{\textbf{Answer: A}}$

30. VOIDED Austin is now observing LARES, but in this problem he is allowed to observe LARES at any declination. Assume that LARES is a uniform spherical ball, and for the purposes of this problem, assume that LARES is a perfect blackbody; i.e. THA-18N has an albedo of 0. What is the brightest apparent magnitude Austin can observe LARES? You can use the fact that the Sun's apparent magnitude is -26.74.

   (a) 6.78
   (b) 7.77
   (c) 8.76
   (d) 9.75
   (e) 10.74

Solution:

The shortest distance Austin can be from LARES is simply its orbiting distance (when LARES is directly overhead Austin):

$$d = 1450 \text{ km}.$$

Let the radius of LARES be r. Power LARES receives from the Sun:

$$P = \frac{L_\odot}{4\pi(1\text{AU})^2}\pi r^2$$

Flux LARES emits to Boston:

$$F = \frac{P}{4\pi d^2} = \frac{L_\odot}{16\pi}\left(\frac{r}{1\text{AU}\cdot d}\right)^2$$

Magnitude-Flux relation with the Sun:

$$m_\odot - m = -2.5\log\left(\frac{F_\odot}{F}\right) = -2.5\log\left(\frac{\frac{L_\odot}{4\pi(1\text{AU})^2}}{\frac{L_\odot}{16\pi}\left(\frac{r}{1\text{AU}\cdot d}\right)^2}\right) = -2.5\log\left(\frac{4d^2}{r^2}\right) \implies$$

$$m = 7.77$$

$\boxed{\textbf{Answer: B}}$