2023 National Astronomy Competition

1 Instructions (Please Read Carefully)

The top 5 eligible scorers on the NAC will be invited to represent USA at the next IOAA. In order to qualify for the national team, you must be a high school student with US citizenship or permanent residency.

This exam consists of 3 parts: Short Questions, Medium Questions and Long Questions.

The maximum number of points is **240 points**.

The test must be completed within 2.5 hours (150 minutes).

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific or graphing calculator for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on April 1st.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam. Exam is proctored and recorded. You are expected to have your video on at all times.

We acknowledge the following people for their contributions to this year's exam:

Wesley Antônio Machado Andrade de Aguiar, Erez Abrams, Lucas Pinheiro, Sahil Pontula, Joe McCarty, Hagan Hensley, Leo Yao, and Andrew Liu.

2 Short Questions - 35 points

1. **(10 points)** White Dwarfs are stars in their last stage of life that are prevented from collapsing only by the electron degeneracy pressure. This pressure is an outward one exerted by the electrons inside the star, which are fermions subject to the Pauli exclusion principle. We can find its value by the following formula, which is derived from the theory of fermion gases:

$$p_{\text{electron}} = \frac{2}{3}u = \frac{\hbar^2}{5m_e}(3\pi^2)^{2/3}n^{5/3}$$

where n is the number density of electrons in the star. This pressure balances the inward gravitational pressure, which is given by

$$p_{\text{grav}} = -\frac{\Omega}{3V}, \quad \Omega = -\frac{3GM^2}{5R}$$

where Ω is the value of the total potential energy of the star.

   (a) **(8 points)** If the star contains nuclei with atomic number Z and mass number A, what is the density value of the white dwarf in function of its total mass M, A, Z, and other fundamental constants?

   (b) **(2 points)** Find what is the value of k in the relation $M \propto V^k$, where V is the volume of the star.

Solution:

(a) To obtain the electron pressure in function of the density, we can use that $n = N_e/V = (ZN)/V$, where N is the number of atoms in the star. Since we can consider that just the protons and the neutrons contribute to the mass of an atom, we can say that $N = M/(Am_p)$, and so $n = \dfrac{Z\rho}{Am_p}$. Besides that, since $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$,

$$R^3 = \frac{3M}{4\pi\rho} \Rightarrow R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$$

Therefore, we can balance the electron and the gravitational pressure (the condition for the existence of the white dwarf) to obtain the value of the density:

$$p_{\text{electron}} = p_{\text{grav}} \Rightarrow \frac{\hbar^2}{5m_e}(3\pi^2)^{2/3}\left(\frac{Z\rho}{Am_p}\right)^{5/3} = \frac{3GM^2}{20\pi R^4} = \frac{G}{5}\left(\frac{4\pi}{3}\right)^{1/3} M^{2/3}\rho^{4/3}$$

$$\Rightarrow \boxed{\rho = \frac{4G^3M^2m_e^3}{27\pi^3\hbar^6}\left(\frac{Am_p}{Z}\right)^5}$$

(b) From the last item, we see that $\rho \propto M^2$. So, since $\rho = MV^{-1}$, $MV^{-1} \propto M^2 \Rightarrow M \propto V^{-1}$. Therefore, $\boxed{k = -1}$, and MV is a constant for white dwarfs.

2. **(10 points)** The Large Magellanic Cloud (LMC) is a galaxy with a redshift of $z = 8.75 \times 10^{-4}$.

   (a) **(4 points)** What is the radial velocity of the LMC with respect to the Solar System? Is is getting closer or farther from the Solar System?

(b) **(4 points)** Hubble's Law is a well-known method of calculating the distance to a galaxy. Using this approach, calculate the distance between the LMC and the Solar System.

(c) **(2 points)** Is Hubble's Law a reasonable method to determine the distance to the LMC? Explain your answer.

Solution:

(a) It is possible to use the following expression to determine LMC's radial velocity:

$$z = \frac{v}{c}$$

$$v = cz$$

$$v = 2.998 \times 10^8 \times 8.75 \times 10^{-4}$$

$$v = 2.62 \times 10^5 \text{ m/s}$$

Since the redshift is positive, the galaxy is getting farther from the Solar System.

(b) Using Hubble's Law:

$$v = H_0 d$$

$$d = \frac{v}{H_0}$$

$$d = \frac{2.62 \times 10^5}{70 \times 10^3}$$

$$d = 3.7\ Mpc$$

(c) Hubble's Law is only valid for distant galaxies. The redshift for the LMC is extremely low, which indicates that it is a close galaxy. The problem of using Hubble's Law with nearby galaxies is that gravitational effects have a significant impact on the velocities, which makes a calculation that only takes into account the velocity due to the expansion of the Universe imprecise. For distant galaxies, these effects are negligible compared to the velocity that results from the expansion of the Universe, but this is not the case for the LMC.

3. **(15 points)** *Culmination Time*

(a) **(12 points)** In Lubbock, Texas ($\lambda = 101^\circ\,53'\,\mathrm{W}$, $\phi = 33^\circ\,35'\,\mathrm{N}$) on September 22nd, what is the local time of upper culmination of Vega ($\alpha = 18\,\mathrm{hr}\,37\,\mathrm{min}$, $\delta = 38^\circ\,47'$)? The time zone of Lubbock is CDT, UTC$-$5. Assume that the equation of time is 6 min at the relevant time, in the convention of solar time minus mean time.

(b) **(3 points)** Name the two primary factors which contribute to the equation of time, and give a brief one-sentence explanation for why each causes solar time to differ from mean time.

Solution:

(a) September 22th is the autumnal equinox, so the right ascension of the Sun is 12 hr. **(2 point)**

By $\Theta = h + \alpha$, where Θ is the local sidereal time, h is the hour angle, and α is the right ascension, the local sidereal time at upper culmination of Vega (hour angle 0) is $\Theta = \alpha_{\mathrm{Vega}}$, so the hour angle of the Sun is $\alpha_{\mathrm{Vega}} - 12\,\mathrm{hr}$ and the local solar time is α_{Vega}. **(3 points)**

Applying the equation of time (ET), mean solar time is $\alpha_{\mathrm{Vega}} - ET$. **(2 points)**

Greenwich mean solar time is therefore $\alpha_{\mathrm{Vega}} - ET - \lambda$, using the convention of longitude as positive if east. In order to convert time zones, we must subtract 5 hr. (Note that UTC and GMT are not exactly identical, but they agree to within a second, so the difference is irrelevant here.) **(3 points)**

Performing the calculations,

$$18\,\mathrm{hr}\,37\,\mathrm{min} - 6\,\mathrm{min} + 101^\circ\,53' - 5\,\mathrm{hr} = \boxed{20{:}19}.$$

(2 point)

(b) The two primary factors are: **(1.5 point each - 0.5 point for naming the effect and another 1 point for the explanation)**

- Eccentricity of the Earth's orbit: the actual Sun doesn't move across the sky at a constant angular speed relative to the background stars, unlike the mean Sun.
- Obliquity of the ecliptic: the actual Sun is on the ecliptic, whereas the mean Sun is on the equator; the angular velocity of the Sun isn't always on and parallel to the equator, which contributes further to non-uniformity of the increase in right ascension.

3 Medium Questions - 65 points

1. **(20 points)** *Hyperfine splitting*

   The 21cm spectral line of hydrogen is a result of the interaction between the electron's and proton's quantum mechanical spin (known as hyperfine splitting). The spins can be either aligned or anti-aligned. When a hydrogen atom decays from the higher energy state to the lower energy state, a photon is emitted with energy equal to the energy difference in these two states.

   The intrinsic magnetic moment of the electron μ_e is approximately equal to the Bohr magneton $\mu_B = \frac{e\hbar}{2m_e}$, and the intrinsic magnetic moment of the proton μ_p is roughly $2.8\frac{e\hbar}{2m_p}$. Electrons and protons (and any particles with spin $\frac{1}{2}$) have permanent magnetic dipole moments $\vec{m}$ with magnitude equal to the intrinsic magnetic moment.

   (a) **(5 points)** Consider two classical magnetic dipoles $\vec{m}_1, \vec{m}_2$ separated by a distance $\vec{r} = r\hat{r}$. The magnetic field from a perfect magnetic dipole $\vec{m}$ is

   $$\vec{B} = \frac{\mu_0}{4\pi}\frac{3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m}}{r^3}$$

   The potential energy of a magnetic dipole in an external magnetic field $\vec{B}$ (e.g. the field created by the other dipole) is $U = -\vec{m}\cdot\vec{B}$.
   Using these two formulas together, what is the energy U_{int} of the interaction between two magnetic dipoles $\vec{m}_1, \vec{m}_2$ separated by a distance $\vec{r} = r\hat{r}$?

   (b) **(6 points)** In our case, we can assume that r is the Bohr radius a_0, and both $\vec{m}_1$ and $\vec{m}_2$ are oriented perpendicularly to $\vec{r}$. What is the interaction energy U_{int} between the magnetic moments of the proton and the electron? Write your answer in terms of $\vec{m}_p \cdot \vec{m}_e$.

   (c) **(2 points)** Which state is the lower energy state: the state where the spins are aligned, or the state where the spins are anti-aligned?

   (d) **(7 points)** The energy of the photon emitted by the transition between these two states is equal to the energy difference in the two states. What value would you predict for the wavelength of the 21cm spectral line? (You should get the right answer to within an order of magnitude, but note that we are treating a quantum mechanical system classically and so there will be large errors.)

Solution:

(a) The interaction energy between the pair of magnetic dipoles is equal to the potential energy of one of the dipoles due to the other dipole's magnetic field. We can write this as

$$U_{\text{int}} = -\vec{m}_1\cdot\vec{B}(\vec{m}_2) = -\vec{m}_1\cdot\frac{\mu_0}{4\pi}\frac{3\hat{r}(\vec{m}_2\cdot\hat{r}) - \vec{m}_2}{r^3}$$

$$= -\frac{\mu_0}{4\pi}\frac{3(\vec{m}_1\cdot\hat{r})(\vec{m}_2\cdot\hat{r}) - \vec{m}_1\cdot\vec{m}_2}{r^3}$$

It's good to check - this expression is symmetrical in $\vec{m}_1$ and $\vec{m}_2$, so it didn't matter which of the dipoles we used to do the calculation.

(b) Since $\vec{m}_1$ and $\vec{m}_2$ are perpendicular to $\vec{r}$, the $\vec{m}\cdot\vec{r}$ terms vanish and the expression simplifies dramatically. If we plug in a_0 for the radius r, then the interaction energy becomes:

$$U_{int} = \frac{\mu_0}{4\pi a_0^3}\vec{m}_p\cdot\vec{m}_e$$

(c) $\vec{m}_p \cdot \vec{m}_e$ is positive if the spins are aligned and negative if they are anti-aligned. Since U_{int} is proportional to $\vec{m}_p \cdot \vec{m}_e$, we can see that the higher energy state has the spins aligned while the lower energy state has the spins anti-aligned.

(d) The magnitude of $\vec{m}$ for the electron or proton is just equal to the intrinsic magnetic moment. This means that the aligned state has interaction energy

$$E_+ = \frac{\mu_0}{4\pi a_0^3}\mu_e\mu_p$$

and the anti-aligned state has interaction energy

$$E_- = -\frac{\mu_0}{4\pi a_0^3}\mu_e\mu_p$$

The energy of the emitted photon is equal to the energy difference in these two states, which is

$$\Delta E = E_+ - E_- = 2\frac{\mu_0}{4\pi a_0^3}\mu_e\mu_p = \frac{\mu_0}{4\pi a_0^3}\frac{2.8(e\hbar)^2}{2m_e m_p} = 1.1 \cdot 10^{-6}\text{eV}$$

The energy of a photon is $E = \frac{hc}{\lambda}$, so the corresponding wavelength of this photon is 112cm. (As it turns out, this is off from the correct value by a factor of exactly $\frac{16}{3}$, which can only be explained by quantum mechanics. Specifically, the wavefunctions of the nucleus and electron actually overlap, whereas we assumed they were discrete point particles with a well-defined separation.)

2. **(20 points)** Astrophysics studies both the smallest and largest scales of physics. The latter is one of the main focuses of cosmology. Here, we look into the former - the role of quantum mechanics in the astrophysics of stars. For this problem, assume that the Sun's core has a proton number density of $n_c \approx 6 \times 10^{31}$ m^{-3} and temperature $T_c \approx 15$ million K.

   (a) **(2 points)** Suppose that two hydrogen nuclei (protons) are flying towards each other in equal and opposite directions, with an impact parameter (distance of closest approach) of $d \approx 1$ fm. If all of this initial kinetic energy came from the average thermal energy of an ideal gas, what would be the requisite temperature $T_{\text{classical}}$ for "fusion" to occur?

In quantum mechanics, particles are described by wavefunctions ψ, whose squared norms $|\psi|^2$ govern the probability of finding the particle in a particular state (e.g., having a certain position, momentum, or energy). Particles are said to behave as waves with a wavelength given by the de Broglie wavelength $\lambda = h/p$, where p is the particle's momentum and h is Planck's constant. It's also convenient to define $p = \hbar k$, where k is the wavevector and $\hbar = h/(2\pi)$.

   (b) **(2 points)** Evaluate the de Broglie wavelength λ_{dB} in fm (1 fm $= 10^{-15}$ m) for a proton with the average thermal energy of an ideal gas at the temperature of the Sun's core, T_c.

In Figure 1, we see that it's classically impossible for a nucleus to make it through the potential barrier with $U_0 > E$ (potential energy larger than the total energy). However, quantum mechanics gives a nonzero probability to "tunnel" through the barrier. To solve for ψ in the regions before, inside, and after the barrier, we can use an ansatz $\psi(x) = Ae^{ik(x)x}$, where A is a normalization constant and $k(x) = \sqrt{\frac{2m}{\hbar^2}(E - V(x))}$.

   (c) **(2 points)** In which regions of Figure 1 is k purely real? Purely imaginary?

   (d) **(2 points)** Modeling the Sun's main fusion reaction as a single process where two protons and two neutrons combine to form a He-4 nucleus, calculate how much energy is released in a single fusion reaction.

(e) **(6 points)** Not every proton-proton collision results in fusion. Calculate the probability P_{fusion} of fusion per proton necessary to sustain the Sun's current luminosity. You may assume 1% of the Sun's mass is comprised of protons available for fusion. For this part, treat the fusion reaction as a collision between two protons (i.e. ignore the neutrons). Assume the cross section for collision is given by $\pi\lambda_{\text{dB}}^2$ where λ_{dB} is the de Broglie wavelength corresponding to being in thermal equilibrium at temperature T_c.

(f) **(6 points)** Consider one proton (moving) in the potential of the other (at rest) in the simplified 1D model of Figure 1. Suppose that U_0 is given by the Coulomb repulsion energy of the two protons at a distance $d \approx 1$ fm and that b is the approximate width of the barrier. Here, b is the impact parameter for a proton repelled by the Coulomb force with initial kinetic energy (outside the barrier) equal to the average thermal energy E. Assuming the potential energy $V(x) = 0$ outside the barrier, find the temperature T_{quantum} such that the probability that protons can tunnel through the barrier and fuse is P_{fusion}.

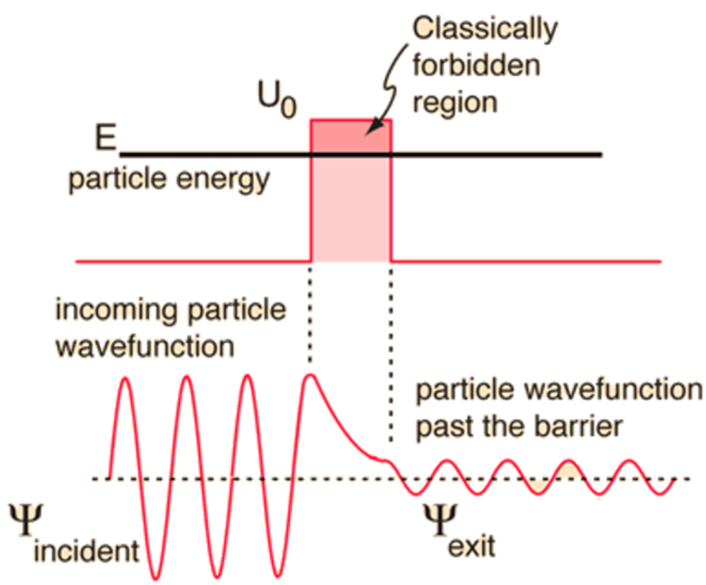


Figure 1: Quantum tunneling.

Solution:

(a) We have $2K = 3k_B T_{\text{classical}} = \frac{e^2}{4\pi\epsilon_0 d}$, from which $T_{\text{classical}} \approx 5.6 \times 10^9$ K.

(b) We have $\frac{3}{2}k_B T_c \approx \frac{(h/\lambda_{\text{dB}})^2}{2m_p}$, from which $\lambda_{\text{dB}} \approx 650$ fm.

(c) Before and after the barrier k is purely real, inside the barrier it is imaginary.

(d) The mass defect for the simplified proton-proton fusion reaction is $\Delta m = 29.7$ MeV/c^2, so $\Delta E = \Delta mc^2 \approx 4.75 \times 10^{-12}$ J are released per fusion reaction.

(e) The collision rate is given by $R_c = n_c \pi\lambda_{\text{dB}}^2 v_{\text{RMS}} \approx 5 \times 10^{13}$ s^{-1}, where $v_{\text{RMS}} = \sqrt{3k_B T_c/m_p}$. Thus $N = L/\Delta E = 8 \times 10^{37}$ s^{-1} is the number of fusion reactions per second (where L is the Sun's luminosity). Finally, this rate per proton is $N/(0.01M/m_p) \approx 6.74 \times 10^{-18}$ s^{-1}. Thus $P_{\text{fusion}} \approx 10^{-31}$.

(f) Transmission through the barrier goes as $e^{-2\alpha b}$, where $\alpha = \sqrt{\frac{2m}{\hbar^2}(U_0 - E)}$ is the imaginary wavevector, $U_0 \approx 2.31 \times 10^{-13}$ J is given by the Coulomb repulsion energy at $d \approx 1$ fm, E is given by the average thermal kinetic energy, and $b = e^2/(4\pi\epsilon_0 E)$ is the width of the well. Let P denote the probability for fusion. Then,

$$P = \exp\left(-2\sqrt{\frac{2mU_0}{\hbar^2}}\frac{e^2}{4\pi\epsilon_0 E}\right).$$

Here, we have made the assumption that $U_0 \gg E$, whose validity we can check after solving. With $P = 10^{-31}$ and $E = \frac{3}{2}k_B T$, $T_{\text{quantum}} \approx 80$ million K. Note that $T_{\text{quantum}}/T_{\text{classical}} \approx E/U_0 \approx 0.01$, so our assumption is valid.

Note: A more rigorous solution to this problem would involve modeling the potential more realistically than a square well. For example, approaches include modeling long-range potentials as power laws or even just effective square wells (though much more rigorously than we have done here). Furthermore, the physics is more interesting (and realistic) in 3D, where considerations of angular momentum and effective potentials become important. Common methods in quantum mechanics such as the WKB method are used in problems like this and, in general, one would need to carefully integrate over the barrier to derive the transmission coefficient.

3. **(25 points)** *Into the Wilds*

You are an astronaut exploring uncharted parts of space in your trusty spaceship, when suddenly you fall into a magical portal to another universe!

Looking for a way back home, you fly to the nearest solar system. Just like normal solar systems, it is home to several planets in orbit around a star - except with one minor difference: it's miniature! The star is only 4 kilometers in diameter, and the whole solar system could fit within a small country back home on Earth.

You know your astrophysics well enough to know that this should be impossible! The only explanation is that some law of physics works differently in this universe. Since you're still alive, chemistry must be unchanged, so quantum mechanics and electromagnetism must work the same as you're used to. Therefore, the only thing that can be different is gravity.

You decide to do some experiments around the solar system to figure out exactly what is different about gravity here.

(a) **(10 points)** First you measure the orbital periods t_i and orbital semi-major axes a_i of the planets in the solar system. You name the planets P_1 through P_5, because you are incredibly unoriginal. Your data is as follows:

Planet	a (km)	t (minutes)
P_1	5.1	1:51
P_2	9.0	4:10
P_3	12.1	6:38
P_4	17.0	11:02
P_5	20.3	14:46

Based on this data, do you think that gravity here follows an inverse square law? Justify your answer with calculations and/or drawing an appropriate plot. (You needn't be too formal with statistics here, any well-reasoned argument will get full credit.)

(b) **(3 points)** If the mass of the star is M and the gravitational constant in this universe is G', then what is $G'M$?

This is a good start! Unfortunately, without a known mass, you can't determine the gravitational constant G'. You decide to give the planets a closer look to see what else you can find out. You'll start from the outermost and work your way in.

(c) **(5 points)** After narrowly escaping being eaten by P_5's local fauna, you penetrate the stormy atmosphere of P_4 to find that it's made almost entirely of water ($\rho = 997 \frac{\text{kg}}{\text{m}^3}$) with a radius of 500 m. While standing in your ship floating on the water's surface, you experience a heavy gravitational pull - 2 times what you're used to on Earth.

What is the gravitational constant in this universe?

(d) **(2 points)** What is the mass of this solar system's star?

(e) **(5 points)** You notice that this star looks yellowish-orange, suggesting its surface temperature is a little lower than that of the Sun. Taking a spectrum, you measure that this star's spectral radiance peaks at a wavelength of 916 nm.

The lifetime of a star can be roughly estimated based on its luminosity and the mass of fuel it has available to burn*. Use a simple scaling argument to roughly estimate the lifetime of this solar system's sun. Explain your reasoning.

(Hint: use our Sun's lifetime of 10^{10} years as a reference point.)

*A more careful calculation shows that the temperature at the core of the star wouldn't actually be enough to ignite fusion, so nuclear physics must work differently here in order for fusion to be possible. Let's ignore that for the sake of this question, though.

Solution:

(a) If gravity follows an inverse square law, then Kepler's third law should hold. This means that $\frac{t^2}{a^3}$ should be a constant for objects orbiting around the same mass.

Planet	a (km)	t (minutes)	$\frac{t^2}{a^3}$ (10^{-2} min^2 km^{-3})
P_1	5.1	1:51	2.58
P_2	9.0	4:10	2.38
P_3	12.1	6:38	2.48
P_4	17.0	11:02	2.47
P_5	20.4	14:46	2.57

The mean value is $\frac{t^2}{a^3} = 2.50 \cdot 10^{-2}$ min^2 km^{-3}. The standard deviation of these values is $0.08 \cdot 10^{-2}$ min^2 km^{-3}, which is comparatively small and probably just the result of random measurement errors. This means that the data supports $\frac{t^2}{a^3}$ being constant, which implies that gravity follows an inverse square law.

(You can do much more careful statistics with this data, but any well-reasoned argument can get full credit.)

Alternative solution: if Kepler's 3rd law is true, then plotting $\log(a)$ against $\log(t)$ should show a linear relationship with a slope of $\frac{3}{2}$. You can just draw the graph and show it visually. If you were to perform a linear regression, the slope of the line is 1.499 ± 0.017, which is in agreement with the expected value of $\frac{3}{2}$.

(b) From Kepler's third law, we can write

$$\frac{t^2}{a^3} = \frac{4\pi^2}{G'M}$$

In the previous part, we found the value of $\frac{t^2}{a^3}$ to be approximately $\frac{t^2}{a^3} = 2.50 \cdot 10^{-2}$ min^2 km^{-3}.

Thus, $G'M = 4\pi^2 \frac{a^3}{t^2}$

$$= 4.39 \cdot 10^8 \mathrm{m}^3\mathrm{s}^{-2}$$

(c) The mass of P_4 is just

$$m = \frac{4}{3}\pi r^3 \rho = 5.22 \cdot 10^{11} \text{ kg}$$

The gravitational acceleration at the surface is $a = \frac{G'm}{r^2}$. We can invert this to solve for G': $G' = \frac{ar^2}{m}$. The measured gravitational acceleration is $a = 2(9.81 \text{ m s}^{-2})$.

This means that the value of G' is $9.4 \cdot 10^{-6}$ m^3 kg^{-1} s^{-2}.

(d) We found $G'M$ and now we have G', so calculating M is straightforward:

$$M = \frac{G'M}{G'} = 4.7 \cdot 10^{13} \text{ kg}$$

(e) Note that you can't use empirically derived scaling relations for the lifetime of a star (e.g. $T \propto M^{-2.5}$) since G' is different!

The scaling argument just comes from conservation of energy: a star's life ends when it burns through all its fuel, and the rate of burning fuel is proportional to its luminosity. Therefore a star's lifetime scales roughly as its mass divided by its luminosity.

For this star, we can calculate the luminosity using Wien's law and the Stefan-Boltzmann law:

$$L_{star} = 4\pi r^2 \sigma T^4 = 4\pi r^2 \sigma \left(\frac{2898 \mu\text{m K}}{916\text{nm}}\right)^4 = 3.23 \cdot 10^{14}\text{W}$$

So the lifetime is

$$T_{star} = T_\odot \frac{M_{star}}{L_{star}} \frac{L_\odot}{M_\odot} = 3 \cdot 10^5 \text{ years.}$$

Any answer around this value is fine due to rounding.

Most data from this problem was taken from the wonderful 2019 video game *Outer Wilds*.

4 Long Questions - 140 points

1. **(45 points)** *The Sundial II*

After gaining an understanding of a sundial, and how the path of the shadow is a straight line on the equinoxes running from due West to due East, Leo now wants to create his own.

At 6am on March 20th, as the Sun is rising, Leo, who is at $(40°\text{N}, 75°\text{W})$, plants a (straight) stick vertically on the ground. At that moment, he marks out a (finite) line on the ground in the direction of the shadow of the stick at that moment, labeling it with the current time. Every hour afterwards, on the hour, he marks out a new line in the current direction of the shadow.

For the following two parts, either show a calculation or explain your answer for credit. Calculating specific values is not necessary, but may help with later parts.

(10 points)

(a) Is the angle between the lines corresponding to 12pm and 1pm greater than, equal to, or less than $15°$?

(b) Is the angle between the lines corresponding to 5pm and 6pm greater than, equal to, or less than $15°$?

At 6pm, after drawing his last line, the sun sets. Leo starts cleaning up his setup, taking down the stick. To check his calculations against experiment, Leo measures the angles between pairs of lines on the ground. Using a very precise protractor, he gets the following values (rolling sums are also provided):

Lines	Angle (degrees)	From 6am (degrees)
6am-7am	9.00570	9.00570
7am-8am	9.48468	18.49038
8am-9am	10.90026	29.39064
9am-10am	13.56767	42.95832
10am-11am	17.77405	60.73237
11am-12pm	22.66646	83.39883
12pm-1pm	24.81144	108.21027
1pm-2pm	21.90336	130.11363
2pm-3pm	16.95555	147.06918
3pm-4pm	13.00713	160.07630
4pm-5pm	10.58101	170.65731
5pm-6pm	9.34269	180.00000

Notably, he finds that they deviate from the values he expects! After frantically checking his calculations and measurements and finding no discrepancies, Leo suspects that the stick might have been slightly tilted. Unfortunately, he took down the stick already, and so can no longer measure it directly.

To help Leo out, we'd like to reconstruct the parameters of the stick tilt. As you work through the following parts, keep in mind this overall goal; it may help to think through all the steps first before proceeding. Ignore atmospheric refraction and the equation of time.

(35 points)

(c) To start, define a 2-D coordinate system for the ground, where the x-axis points due North and the y-axis points due East. Choose and clearly indicate an appropriate origin.

(d) Choose a specific time during the day when the Sun is above the horizon. In your coordinate system, compute the coordinates of the tip of the shadow. Clearly indicate the chosen time and any assumptions made, and justify assumptions if they are not general.

(e) From these coordinates and other information, find a condition on the location of the other end of the stick. Provide your answer as a linear equation in the form $y = mx + b$ for some m, b.

(f) Determine more conditions as necessary, and solve to find the coordinates of the other end of the stick.

(g) Determine the angle, from the vertical, that the stick was tilted, and in what direction.

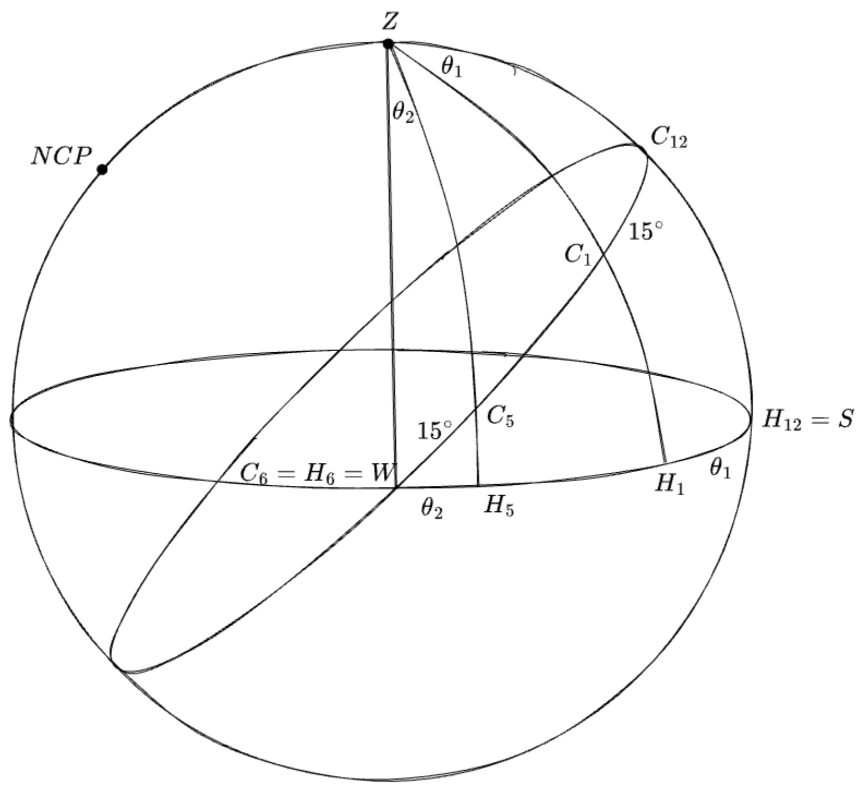
Depending on method chosen, points on subparts may vary. **Due to small values involved, it is recommended to carry calculations to at least 4 decimal places. Answers to the final part will be integers. No credit given for guesses without justification.**

Solution:

(a) The first important observation to make is, because the stick is vertical and perpendicular to the ground, the line of the shadow points in a direction directly opposite the azimuth of the Sun at that moment. Therefore, we can compare angles between lines by comparing changes in azimuth of the Sun between two times.

As March 20th is the spring equinox (which can also be deduced as the problem mentions the Sun rising at 6am and setting at 6pm), the Sun travels along a great circle in the sky at a constant rate of 15° per hour. Let C_{12}, C_1, C_5, C_6 be points along the great circle where the Sun is at 12pm, 1pm, 5pm, 6pm respectively. As the Sun rises at 6am and sets at 6pm, we know solar time corresponds with local time (which is also confirmed as 75°W is perfectly in the center of the UTC-5 time zone). Therefore, the Sun is on the meridian at 12pm and on the horizon due west at 6pm.

Extend the arcs from Z through C_{12}, C_1, C_5, C_6 to meet the horizon at $H_{12} = S, H_1, H_5, H_6 = C_6 = W$. Then changes in azimuth correspond to arcs along the horizon: The arc $H_{12}H_1$ is the angle between 12pm and 1pm and has length θ_1, and the arc H_5H_6 is the angle between 5pm and 6pm and has length θ_2. The arc lengths θ_1, θ_2 also are the same as the corresponding angles from Z.



As one possible solution, consider the triangle $ZC_{12}C_1$. As $\angle ZC_{12}C_1 = 90^\circ$, then by the spherical law of sines:

$$\frac{\sin\theta_1}{\sin 15^\circ} = \frac{\sin 90^\circ}{\sin ZC_1}$$

But as $\sin 90^\circ = 1$ and $ZC_1 < 90^\circ$, we have that the right-hand side is greater than 1 and therefore $\theta_1 > 15^\circ$.

Many other solutions involving spherical triangles are possible (arguments from other parts of the solution can also be adapted to this part).

(b) As one possible solution, consider the triangle C_5H_5W. As $\angle C_5H_5W = 90°$, then by the spherical law of cosines:

$$\cos C_5H_5 \cos\theta_2 = \cos 15°$$

But as $\cos C_5H_5 < 1$, we then have $\cos\theta_2 > \cos 15°$ and therefore $\theta_2 < 15°$.

Again, other choices of spherical triangles leading to a correct solution are possible.

(c) From this point onwards, there are two similar but slightly different ways to solve the problem, depending on choice of origin. Points on subparts differ between the methods, but the overall amount of work needed is about the same. We will be calling these methods "Tip of stick" and "Base of stick".

Tip of stick: Drop a perpendicular from the tip of the stick to the ground, and set the point directly below the tip of the stick to be the origin.

While this definition initially seems more contrived, note that the position of the tip of the shadow depends only on the position of the tip of the stick, and the Sun. Therefore, we can directly write down positions in our coordinate system of tips of shadows.

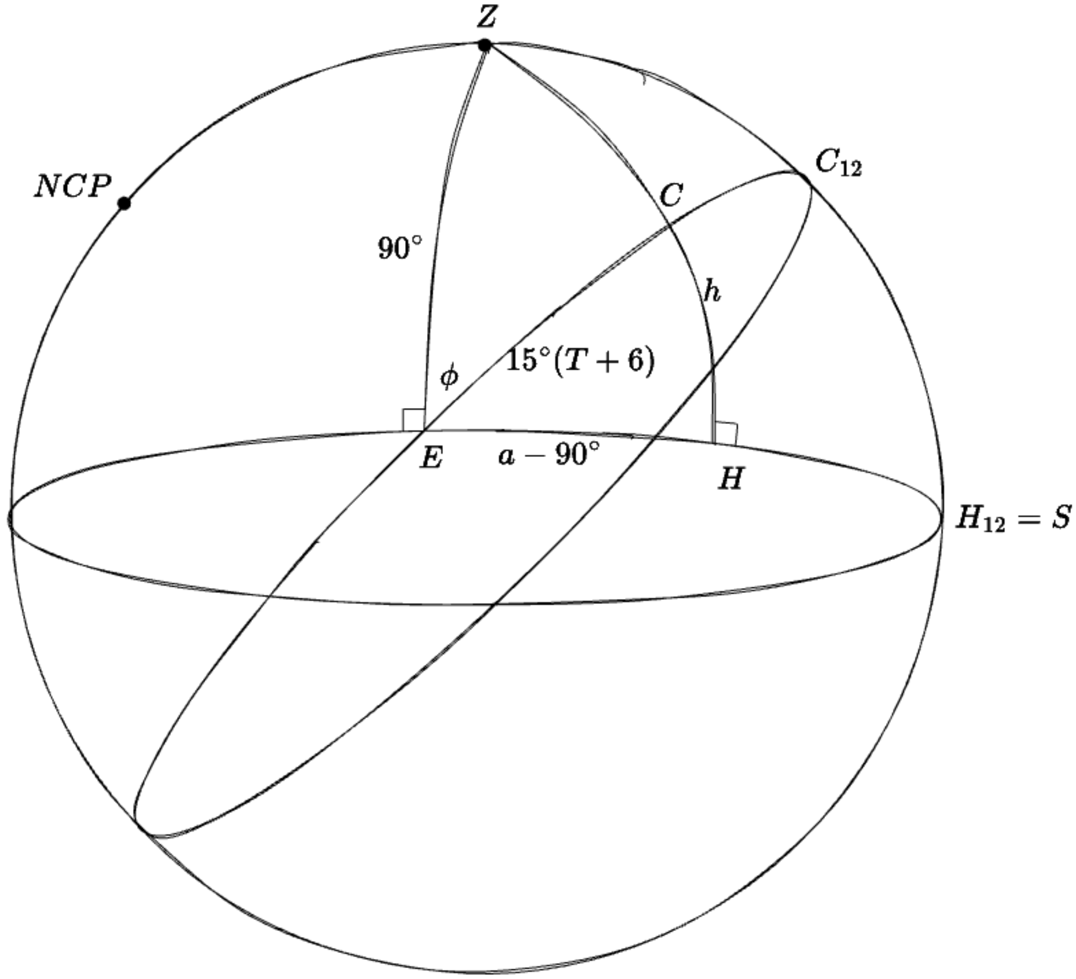
Base of stick: Set the base of the stick to be the origin.

This may be the more immediately apparent choice, and has the advantage that we immediately know what directions shadows point in. However, this method is more difficult to follow through with; an astute observation is needed later to make progress.

(d) **Tip of stick:**

In order to determine lengths of shadows of the ground, we first need to know the height of the stick tip. Therefore, introduce a variable h_0 for the height of the tip of the stick above the ground.

We will carry through h_0 for the rest of the solution. However, note that the problem is invariant under rescaling (no specific lengths are given, and h_0 divides out at the end). Therefore, it is valid to set h_0 to an arbitrary positive value, as long as it is divided out at the end.



Start by computing the altitude and azimuth of the Sun at a given time. For some time T hours after noon, where $T \in (-6, 6)$: Let C be the position of the Sun (along the celestial equator) at the time. Then, as the Sun passes 15° of angle per hour, we have:

$$EC = 15^\circ(T + 6)$$

As angle $\angle ZEC$ subtends arc $ZC_{12} = \phi$ where $EC_{12} = ZE = 90^\circ$, we must have $\angle ZEC = \phi$. Then, we can apply the spherical law of cosines on triangle ZEC:

$$\cos ZC = \cos EZ \cos EC + \sin EZ \sin EC \cos \angle ZEC$$

To get the angle of arc ZC. Then, the altitude is just 90° minus this angle:

$$h = 90^\circ - ZC$$

To find the altitude, use spherical triangle ECH, where we extend ZC to meet the horizon at H. As $\angle CHE$ is right, the $\sin a \sin b \cos C$ term in the spherical law of cosines disappears, so we just have:

$$\cos EC = \cos CH \cos EH$$

Then:

$$\cos EH = \frac{\cos EC}{\cos EH} = \frac{\cos 15^\circ (T+6)}{\cos h}$$

And as EH is the arc along the horizon starting from an azimuth of 90°, we have $a = 90^\circ + EH$.

As the data given in the problem is given for every hour of the day, any integer choice of T where the Sun is above the horizon ($T \in \{-5, -4, \ldots, 5\}$) is valid. As one example, evaluating these equations through for $T = 1$ hour after noon gives $(h, a) = (47.72648^\circ, 202.62906^\circ)$.

If the stick were perfectly vertical, the base of the stick would be at the origin, and the length of the shadow on the ground would be given by:

$$l = \frac{h_0}{\tan h}$$

where h_0 is the height of the stick tip, and h is the altitude of the Sun.

For the specific coordinate axes given, where the x-axis points due North and the y-axis points due East, azimuths correspond perfectly with angles from the x-axis, first rotating towards the y-axis. Therefore, to convert between azimuths and (cartesian) coordinates, no angle offset is necessary.

Then, as the shadow is directed opposite the Sun, the coordinates of the shadow tip would be:

$$\text{(origin coordinates)} + \text{(displacement vector of shadow)} = (-l \cos a, -l \sin a)$$

But this holds regardless of where the base of the stick is! Then we have, for the coordinates of the shadow tip:

$$-\frac{h_0}{\tan h}(\cos a, \sin a)$$

Which for $T = 1$ hour after noon gives:

$$(0.83910, 0.34978)h_0$$

Base of stick:

As the shadow is a projection of a line (the stick) onto the plane of the ground, it must be a straight line. In addition, the line of the shadow must contain the base of the stick (which is on the ground) and the tip of the shadow.

Again, by a rescaling argument, we introduce a length l_0 of a shadow on the ground. The shadow on the ground has an angle given by the data: as the 6am line points due west, the azimuth of the shadow is given by the angle from 6am in the last column of the table, less 90 degrees. Then for an angle from 6am measurement θ, the shadow coordinates are:

$$(l_0 \cos(\theta - 90^\circ), l_0 \sin(\theta - 90^\circ))$$

Then for $T = 1$ hour after noon gives:

$$(0.94992, 0.31251)l_0$$

(e) **Tip of stick:**

Again, shadows are straight lines on the ground having an angle given by the data. Just as azimuths can be converted directly to polar coordinates, they can also be converted directly to slopes:

$$m = \tan(\theta - 90^\circ)$$

Then, to find the intercept b, we just need to substitute in our known coordinate of the shadow tip (x_0, y_0):

$$y_0 = mx_0 + b \implies b = y_0 - mx_0$$

Values for $T = 1$ hour after noon:

$$m = 0.32898, b = 0.07373h_0$$

Base of stick:

To work towards finding the actual tilt, consider an ideal stick, with the same tip, but standing perfectly vertical. Then the ideal shadows cast would have the same tip as our stick, but would be at a different angle on the ground. These shadows would intersect at a different point, the base of the ideal stick. As the base of the ideal stick is directly below the tip of the ideal (and our) stick, this gives us the means to find the tip of the stick in our coordinates.

We therefore want the angle of the ideal shadows. However, this is just an azimuth calculation (described in the previous part)! Therefore:

$$m = \tan a$$

$$y_0 = mx_0 + b \implies b = y_0 - mx_0$$

Values for $T = 1$ hour after noon:

$$m = 0.41685, b = -0.08347l_0$$

(f) **Tip of stick:**

In order to solve for the coordinates (x, y), we need a second condition. At this point, we could just choose another time during the day, compute altitude and azimuth using the same process, and find the coordinates of another shadow tip. However, there are some clever choices and ideas that we can use to avoid fully running through another calculation.

Consider the position of the Sun at noon, or $T = 0$. Then the Sun is on the meridian, implying that it is due South. As it is also still on the celestial equator, we can immediately write down horizontal coordinates for the Sun:

$$(h, a) = (90^\circ - \phi, 180^\circ)$$

Again converting this to shadow tip coordinates:

$$-\frac{h_0}{\tan h}(\cos a, \sin a) = (0.83910, 0)h_0$$

The y-coordinate is 0 as expected, as the shadow should have no deviation in the East-West direction when the Sun faces due South. However, note that the x-coordinate is the same as before! It turns out (and is noted at the start of the problem text) this holds true for any choice of T: the tip of shadows trace out a straight line running West to East on the equinoxes! (for more details as well as a proof, see last year's second round problem with a very similar name, where this was the core concept tested!)

Then finding the slope and line for $T = 0$:

$$m = -0.11572, b = 0.09710h_0$$

Now that we have a pair of linear equations, we can solve for (x, y). As they are already in slope-intercept form, it is easiest to eliminate; doing so and substituting in gives:

$$(x_0, y_0) = (0.05255, 0.09102)h_0$$

Base of stick:

In order to solve for the coordinates (x, y), we need a second condition. Again, we can use the data to get an angle for the shadow on the ground. However, in order to find the coordinates of the tip, we need the length of the second shadow, which we somehow need to relate to the length of the first.

But this is where the "shadow tips draw out a straight line" property proves useful! Regardless of coordinate origin, for coordinate axes aligned the same way, the x-coordinate of the shadow tip remains constant. As we know shadows are drawn from the origin at an angle given by the data, this gives us enough information to find the tip of another shadow.

Taking $x_0 = 0.94992 l_0$ from earlier and taking the Sun's position at noon ($T = 0$):

$$y_0 = x_0 \tan\theta = -0.10993 l_0$$

Then the azimuth of the Sun at noon is simply $a = 180^\circ$, and the ideal shadow runs perfectly North-South. Then, the base of the ideal stick must have the same y-coordinate (no East-West deviation). This gives:

$$y = y_0 = 0 \cdot x - 0.10993 l_0$$

At this point, we can substitute our known value of y into the previous equation, giving the solution:

$$(x, y) = (-0.06347, -0.10993) l_0$$

(g) **Tip of stick:**

We now have the coordinates of the stick base, when the stick tip is at the origin. To get a base to tip vector, we must first invert this vector:

$$(-0.05255, -0.09102) h_0$$

Then, the direction of tilt is just the azimuth of this vector, or $\boxed{240^\circ}$.

To compute the angle o f tilt from the vertical, first compute the length of the deviation on the ground:

$$\sqrt{(0.05255 h_0)^2 + (0.09102 h_0)^2} = 0.10510 h_0$$

Then, as h_0 is the vertical height of the tip, we get the tilt from the vertical:

$$\tan^{-1}\left(\frac{0.10510 h_0}{h_0}\right) = \boxed{6^\circ}$$

Base of stick:

We now have the coordinates of the stick tip, where the stick base is at the origin. Again, we can just take the azimuth of the base to tip vector to get the tilt direction, which gives the same answer of $\boxed{240^\circ}$.

The stick tilt is a bit more difficult, as we don't yet have h_0, the height of the stick; we have l_0, the length of a real shadow. To compute the height of the stick, we first need the length of an ideal shadow at some time. This computation is easiest for noon, or $T = 0$:

$$l_{0,\text{ideal}} = ||(0.94992, -0.10993) l_0 - (-0.06347, -0.10993) l_0|| = 1.01338 l_0$$

Then stick height:

$$h_{0,\text{ideal}} = l_{0,\text{ideal}} \tan(90^\circ - \phi) = 1.20770 l_0$$

Finally, we compute the length of the deviation on the ground:

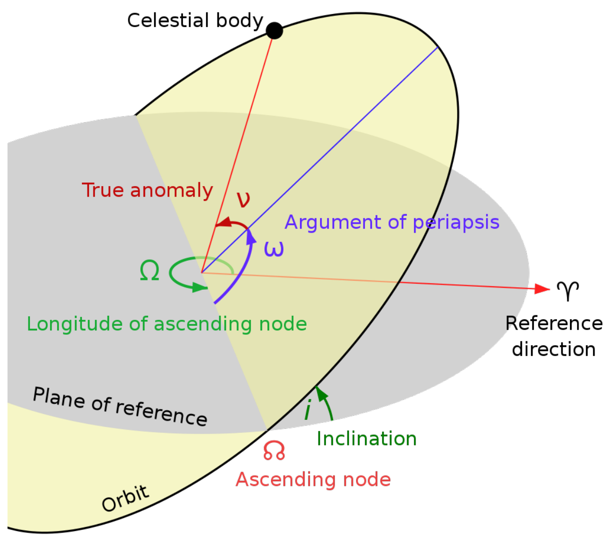
$$\sqrt{(0.06347 l_0)^2 + (0.10993 l_0)^2} = 0.12693 l_0$$

And the tilt from the vertical, which gives the same answer:

$$\tan^{-1}\left(\frac{0.12693 l_0}{1.20770 l_0}\right) = \boxed{6^\circ}$$

2. **(40 points)** A new interstellar comet is about to enter the solar system, and Erez is really excited to calculate some Keplerian elements on its approach and figure out where it's going! The comet, dubbed Scratcher Crab by the astronomical community (for no particular reason), is approaching the solar system with a velocity of 29.8km/s from the direction of Altair, which has equatorial coordinates $\alpha = 19^h 51^m, \delta = 8^\circ 51'$. If the comet were to continue traveling in a straight line with its current velocity, its closest approach to the sun would be at a distance of 1au, and on March 21st, the comet would briefly be in superior conjunction.

   For reference, here is a diagram of orbital elements. Note that the longitude of the ascending node is measured from Aries and the argument of the periapsis is measured from the ascending node:



   (a) **(6 points)** What are the eccentricity and semi-major axis of the orbit?

   (b) **(6 points)** What are the inclination of the orbit and the longitude of the ascending node, both measured with reference to the equatorial plane (not the ecliptic)?

   (c) **(6 points)** What is the angular distance between the incoming and outgoing trajectories of the comet?

   (d) **(6 points)** What is the argument of the periapsis?

   (e) **(6 points)** What are the exit coordinates of the comet (in equatorial coordinates)?

   (f) **(10 points)** Erez wants to check his calculations by making some observations when the comet crosses the plane of the ecliptic. At this time, what is the proper motion of the comet in the sun's frame of reference (not the Earth's)? Give your answer in its equatorial components μ_α, μ_δ.

Solution: Whenever the semi-major axis a appears in this solution, the convention of $a < 0$ for hyperbolas is chosen, but both signs should be accepted as correct. The choice of sign will lead to sign differences in formulae involving a as well. Most answers should have three significant figures, but an additional figure will be included in parentheses in this solution.

(a) There are a few ways we can get the eccentricity and SMA. The first is to use the vis-viva equation to get the specific orbital energy:

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{v^2}{2} = -\frac{\mu}{2a} \implies a = -\frac{\mu}{v^2} = -1.49(5) \cdot 10^{11}\text{m}$$

We can then combine this with the knowledge that the impact parameter equals the semi-minor axis $b = 1\text{au} = 1.49(6) \cdot 10^{11}\text{m}$ and use conic relations to get (noting sign changes because of hyperbola conventions):

$$\begin{aligned} p &= -\frac{b^2}{a} \\ \epsilon &= \sqrt{1 - \frac{p}{a}} \\ &= 1.41(5) \end{aligned}$$

An alternative way to arrive at p is to use the specific angular momentum relation $p = \frac{l^2}{\mu}$ instead as in problem 26 of the first round. Also, instead of doing the eccentricity calculation explicitly, we could note that the problem has been set up so that $p = b = -a$ to immediately arrive at $\epsilon \approx \sqrt{2}$ (this answer, 1.41, and 1.42 should all be accepted as correct).

(b) Orbits are planar. This is a very useful fact, because it allows us to deduce that the orbit should lie in the same plane as the unique plane that contains both the sun and the straight-line trajectory described in the problem. If on the straight-line trajectory, on March 21st the comet is behind the sun (at Aries, since this is the equinox), this means that there is a point in time on the straight-line trajectory where the comet has equatorial coordinates $(0, 0)$. Thus, the orbit lies in the unique plane through the origin, Altair, and Aries. First, this means that the descending node is at Aries, i.e. the longitude of the ascending node is exactly 180°. Second, this allows us to deduce the inclination by four-parts on the spherical triangle with Altair, Aries, and the projection of Aries onto the equatorial plane:

$$\begin{aligned} \cos 90 \cos(-\alpha) &= \sin(-\alpha) \cot \delta - \sin 90 \cot i \\ i &= \cot^{-1}(-\cot \delta \sin \alpha) \\ &= 9.98^\circ \end{aligned}$$

(c) We can use the polar equation of a conic:

$$r = \frac{p}{1 + \epsilon \cos(\phi - \phi_0)}$$

We seek the asymptotes of the hyperbola, so $1 + \epsilon \cos(\phi - \phi_0) = 0$:

$$\begin{aligned} \phi &= \phi_0 + \arccos\left(\frac{-1}{\epsilon}\right) \\ &= \phi_0 \pm 135^\circ \\ \Delta\phi &= 270.^\circ \end{aligned}$$

An answer of 90.° is also acceptable. Alternatively, one could note that a rectangular hyperbola has eccentricity of $\sqrt{2}$, and that is the eccentricity of this hyperbola, so the angle is immediately 90°.

(d) From part (c), the argument of the periapsis is just the argument of incoming trajectory (the argument of Altair) plus 135°. Noting that the argument should be measured from the ascending node, which is at $\alpha = 180°$, we can simply find the great circle distance from $(180°, 0)$ to Altair with Law of Cosines:

$$\begin{aligned}\cos\omega_{in} &= \cos(90-0)\cos(90-\delta) + \sin(90-0)\sin(90-\delta)\cos(180-\alpha)\\ \omega_{in} &= \arccos(\cos\delta\cos(180-\alpha))\\ &= 117°\end{aligned}$$

So the argument of the periapsis is $\omega = 117 + 135 = 252°$. It is quite easy to accidentally measure from the wrong point and get $180° \pm 252°$ or even 197°; these answers have errors which will propagate to the next part as well.

(e) From part (c) again, we know that the argument of the exit point X is $\omega_X = 117° + 270° = 387° > 360°$, which means that the exit point is in the northern hemisphere. Thus, we can use Law of Sines on the spherical triangle with the ascending node, the exit point, and the projection of the exit point onto the equatorial plane:

$$\begin{aligned}\frac{\sin\delta_X}{\sin i} &= \frac{\sin\omega_X}{\sin 90}\\ \delta_X &= \arcsin(\sin\omega_X \sin i)\\ &= 4°34'\end{aligned}$$

And we can use Law of Cosines on the same triangle to get:

$$\begin{aligned}\cos\omega_X &= \cos(\alpha_X - 128)\cos(\delta_X) + \sin(\alpha_X - 180)\sin(\delta_X)\cos 90\\ \alpha_X &= 12^h + \arccos\frac{\cos\omega_X}{\cos\delta_X}\\ &= 13^h 48^m\end{aligned}$$

(f) We can use the conic polar equation to get the orbital distance at the descending node:

$$\begin{aligned}r_{desc} &= \frac{p}{1+\epsilon\cos(180°-\omega)}\\ &= 1.04(8)\cdot 10^{11}\text{m}\end{aligned}$$

From which the velocity perpendicular to the radius is (from angular momentum):

$$\begin{aligned}v_{\perp desc} r_{desc} &= v_\infty b\\ v_{\perp desc} &= \frac{v_\infty b}{r_{desc}}\\ &= 42500\text{m/s}\end{aligned}$$

So the proper motion (angular velocity) is:

$$|\vec{\mu}| = \frac{v_{\perp desc}}{r_{desc}} = 4.059\text{rad/s}$$

Since at the descending node we know the direction of the velocity matches the inclination:

$$\begin{aligned}\mu_\alpha &= |\vec{\mu}|\cos i\\ &= 4.00\cdot 10^{-7}\text{rad/s}\\ &= 0.082\text{as/s}\\ \mu_\delta &= -|\vec{\mu}|\sin i\\ &= -7.03\cdot 10^{-8}\text{rad/s}\\ &= -0.015\text{as/s}\end{aligned}$$

3. **(55 points)** *Finding Friends*

In the future, Earth is perfectly spherical and completely covered with water. Sailor Andrew is on a boat initially positioned at the equator in such that a way that γ, the vernal equinox, is directly overhead. He observes a star χ with right ascension and declination $(\alpha, \delta) = (15°, 10°)$. He wants to chase χ, and he needs your help.

You may assume the following numerical values for each constant:

- $R_\oplus$ is Earth's radius (same as the current day value).
- $\omega = 2\pi/(24 \text{ hrs})$ is Earth's angular frequency.
- His boat is very fast; its maximum speed is $|v| = \omega R_\oplus/4$.
- The vernal equinox γ has celestial coordinates $(0°, 0°)$.
- $(\alpha, \delta) = (15°, 10°)$.

(a) **(2 points)** Calculate the angular distance between γ and χ. Express your answer in degrees.

In his starting position, χ is somewhere above his horizon. As he sails to different parts of the planet and time passes, χ moves around his night sky. He wants sail his boat in order to get χ as close to his zenith as possible. To make sure he is as efficient as possible, he wants to make a few calculations first.

(b) **(2 points)** If he does not move his boat from his starting position, calculate the minimum zenith distance that χ reaches. Express your answer in degrees.

Note: zenith distance is defined as the angular distance to the zenith.

Now, he considers what would happen if he sails straight north at all times.

(c) **(3 points)** Starting from his intial position when γ is overhead, he directs his boat straight north and starts sailing with speed $|v|$. Does χ first cross his north-south meridian or east-west meridian? Briefly explain why this is the case.

(d) **(3 points)** Calculate the zenith distance of χ when it first crosses his north-south meridian. Express your answer in degrees.

(e) **(15 points)** Calculate the zenith distance of χ when it first crosses his east-west meridian. Express your answer in degrees.

Let X be the point in his trajectory (in the celestial sphere) at which χ reaches its minimum zenith distance, and N be the north celestial pole. Note that X is a point in the celestial sphere, and not Earth.

(f) **(6 points)** Argue that $\angle N\gamma X + \angle \chi X N \approx \pi/2$, where both angles are spherical angles. Then, calculate $\angle \chi X N$. Express your answer in degrees.

Note: For this part only, you may assume that the declination of X is small, hence the approximation.

(g) **(15 points)** Calculate the minimum zenith distance that χ reaches. Express your answer in degrees. For this problem:

- You may no longer assume that the declination of X is small. However, you may use the value of $\angle \chi X N$ derived in part (f).
- You are given that the right ascension of X is within $(\alpha \pm 2°)$.
- **His trajectory in the celestial sphere is not necessarily a great circle**. Be careful when computing angles and lengths!

He is not satisfied with these results. He feels there should be an optimal way to get χ directly overhead, as fast as possible. Help him figure out the fastest way to get χ to his zenith.

(h) **(2 points)** Given that he is allowed to maneuver his boat however he wants throughout the duration of his journey, and that he starts from his initial position with γ overhead, show that χ cannot be at his zenith the first time it culminates, i.e., crosses his north-south meridian.

(i) **(7 points)** Let t be the fastest time he can get χ to his zenith. Compute the quantity $(t - 20\text{ hrs})$, in seconds.

Solution:

(a) Let N be the north celestial pole and d be our desired distance. Then, the spherical law of cosines in $\triangle N\gamma\chi$ gives

$$\cos d = \cos 90^\circ \cos(90^\circ - \delta) + \sin 90^\circ \sin(90^\circ - \delta)\cos\alpha,$$

which implies $d = \arccos(\cos\delta\cos\alpha) \approx 17.96^\circ$.

(b) Since χ is to his east, it has not yet culminated. Also, since he is not moving, χ achieves its minimum zenith distance as it culminates.

One way to figure out this minimum zenith distance is to consider the view from his horizon. χ moves through the small circle representing the declination line at δ. Since he is located on the equator, the celestial equator from his perspective passes through the zenith. Therefore, the minimum zenith distance is the distance between the declination line at δ, and the celestial equator, which is $\delta = 10^\circ$.

An alternative approach is to consider his movement through the celestial sphere. Culmination occurs when his location and χ have the same right ascension. At this point, their distance is δ, so this is the minimum zenith distance.

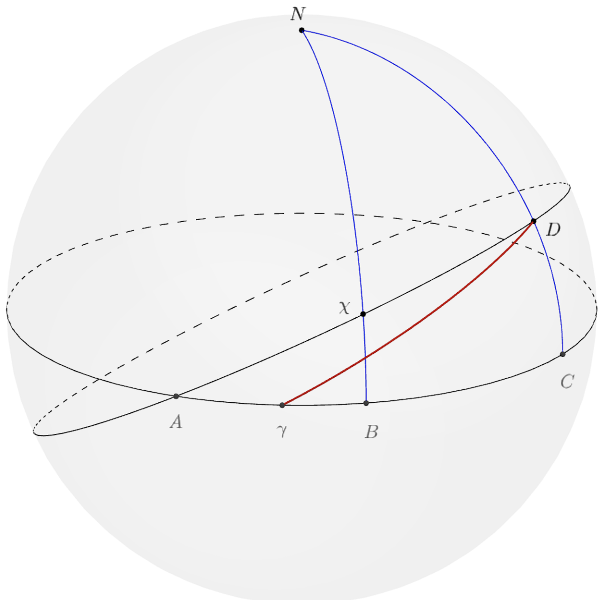
(c) χ starts in the northeastern quadrant of the sky. When χ crosses the north-south meridian, it changes from the east to west hemisphere. When χ crosses the the east-west meridian, it changes from the north to south hemisphere.

At time t, his location in the celestial sphere is given by $(\omega t, \omega t/4)$, so $(\alpha, \alpha/4)$ is a point in his trajectory. Since $\alpha/4 < \delta$, χ culminates in his northern sky. Thus, χ passes the north-south meridian before passing the east-west meridian.

(d) χ culminates when it has the same right ascension as Andrew, so $\omega t = \alpha$. So, the zenith distance is given by

$$\delta - \frac{\alpha}{4} \approx 6.25^\circ.$$

(e) Let D be his position in the celestial sphere. When χ first crosses the east-west meridian, D must satisfy $\angle ND\chi = 90^\circ$, as shown in the diagram below. The red line in the diagram represents his trajectory in the celestial sphere (this trajectory is not a great circle, so we cannot use it in our calculations).



Now, chasing angles:

- The perpendicular condition implies that D is the apex of another great circle (call Γ) that passes through both D and χ. Γ intersects the equator at a point A, such that $DA = 90^\circ$.
- As labelled in the diagram, let B and C be the feet of χ and D from N, respectively (so that $\angle \chi BA = \angle DCA = 90^\circ$). Since D is the apex of Γ, $DA = CA = 90^\circ$.
- We know $\gamma B = \alpha$ and $\gamma C = \omega t$, so $AB = 90^\circ + \alpha - \omega t$.
- We also know that $\chi B = \delta$, and $DC = \omega t/4$.

Our goal is to find the zenith distance, which is χD. To find this arc, we'll need ωt, which will let us solve for the right ascension of D. Let $\varepsilon = \angle \chi AB = \angle DAC$. The four-parts formula in both $\triangle \chi AB$ and $\triangle DAC$ gives

$$\tan \varepsilon = \frac{\tan \chi B}{\sin AB} = \frac{\tan DC}{\sin AC}.$$

Substituting known values, this reduces to the equation

$$\tan \delta = \cos(\omega t - \alpha) \tan\left(\frac{\omega t}{4}\right) \implies \omega t = 4 \arctan\left(\frac{\tan \delta}{\cos(\omega t - \alpha)}\right).$$

We can use this to numerically iterate for ωt. Our initial guess does not need to be good; any value from around 10° to 70° converges in a few iterations, giving us $\omega t \approx 46.97^\circ$.

Finally, a few more angles:

- $\angle \chi ND = \angle BNC = \omega t - \alpha$.
- $N\chi = 90^\circ - \delta$.
- $ND = 90^\circ - \omega t/4$.

These are all known values. The spherical law of cosines in $\triangle N\chi D$ now gives

$$\chi D = \arccos\left(\sin \delta \sin\left(\frac{\omega t}{4}\right) + \cos(\omega t - \alpha) \cos \delta \cos\left(\frac{\omega t}{4}\right)\right) \approx 31.42^\circ.$$

(f) The zenith distance χX is minimized when $\angle \chi X \gamma = 90^\circ$. Let X' be the foot of X from N, so that $\angle NX'\gamma = 90^\circ$. Then, if we let $\beta = \angle NX\chi$:

- N, X, X' lie on the same great circle, $\angle NX\chi + \angle \chi X \gamma + \angle \gamma X X' = 180^\circ$, which implies $\angle \gamma X X' = 90 - \beta$.
- Since the latitude of X is small, and $\triangle XX'\gamma$ is right, we can say that $\angle \gamma X X' + \angle X \gamma X' \approx 90^\circ$, so $\angle X\gamma X' \approx \beta$. This approximation is valid since $\gamma X' = \omega t = 4X'X$, so $\triangle XX'\gamma$ can be approximated by a flat right triangle.
- Since $\angle N\gamma X' = 90^\circ$, β and $\angle N\gamma X$ add to 90°, as desired.

To compute β, we can use our approximation of $\triangle XX'\gamma$ as a flat right triangle to say that $\tan\beta \approx 1/4$, so $\beta \approx 14^\circ$.

(g) Consider $\triangle NX\chi$. We know the lengths of two sides, $N\chi = 90^\circ - \delta$ and $NX = 90^\circ - \omega t/4$, and the values of a nested and adjacent angle, $\angle N = \omega t - \alpha$ and $\angle X = \beta$. We want to find the length of the zenith angle, which is $X\chi$. Unfortunately, we can't directly solve for $X\chi$, but we may first try solving for ωt using the information we already have, which will give us enough information about the triangle to solve for $X\chi$.

Applying the four parts formula:

$$\sin\left(\frac{\omega t}{4}\right)\cos(\omega t - \alpha) = \cos\left(\frac{\omega t}{4}\right)\cot(90^\circ - \delta) - \sin(\omega t - \alpha)\cot\beta.$$

As in part (e), we can isolate ωt and try to solve this equation with numerical iteration:

$$\omega t = 4\arctan\left(\frac{\tan\delta}{\cos(\omega t - \alpha)} - \frac{\sin(\omega t - \alpha)\cot\beta}{\cos(\omega t/4)\cos(\omega t - \alpha)}\right).$$

Unfortunately, this diverges for most values. Luckily, we're given that ωt is within $(\alpha \pm 2^\circ)$, which is a good enough range for us to still find a solution at $\omega t \approx 16.45^\circ$.

Finally, the law of sines in $\triangle X\chi N$ gives

$$\chi X = \arcsin\left(\frac{\cos\delta\sin(\omega t - \alpha)}{\sin\beta}\right) \approx 6.06^\circ.$$

(h) In order for χ to be directly overhead, he needs to have the same declination and right ascension as χ.

The amount of time he needs to increase his latitude to δ is bounded below by

$$\delta/(v/R_\oplus) = \frac{4\delta}{\omega},$$

which is when he travels straight north at all times.

On the other hand, the amount of time it takes him to have the same right ascension is bounded above by

$$\alpha/(\omega - v/R_\oplus) = \frac{4\alpha}{3\omega},$$

which is when he travels straight west at all times. Since $4\alpha/3 < 4\delta$, he cannot get χ to his zenith when it first culminates.

(i) From part (h), he cannot get χ directly overhead the first time it culminates. Therefore, his best strategy is to try to get χ to his zenith the next time it appears above his horizon. After time t, the change in his right ascension due to the rotation of the earth is ωt. Therefore, to optimize his distance travelled, he should travel along a great circle (defined on the surface of the Earth) that increases his longitude by $(2\pi + \alpha - \omega t)$, and increases his latitude by δ.

Let N be the north celestial pole, and Y be his stopping point, which has coordinates $(2\pi + \alpha - \omega t, \delta)$. Note that the coordinates of Y are his longitude and latitude on Earth, and not in the celestial sphere. Also, $Y\gamma = \omega t/4$, since the maximum speed his boat can travel is $\omega R_{\oplus}/4$. Now, the spherical law of cosines in $\triangle NY\gamma$ gives

$$\cos\left(\frac{\omega t}{4}\right) = \cos\delta\cos(\alpha - \omega t),$$

so

$$\omega t = 4\arccos(\cos\delta\cos(\alpha - \omega t)).$$

Now we can numerically iterate. Using $t = 20$ hrs as a first guess, we get $\omega t = 300.2^\circ$, which gives $(t - 20\text{ hrs}) \approx 48$ s. To account for approximation errors, a large range of values in this general vicinity are acceptable.