# 2023 National Astronomy Competition

## 1 Instructions (Please Read Carefully)

The top 5 eligible scorers on the NAC will be invited to represent USA at the next IOAA. In order to qualify for the national team, you must be a high school student with US citizenship or permanent residency.

This exam consists of 3 parts: Short Questions, Medium Questions and Long Questions.

The maximum number of points is **240 points**.

The test must be completed within 2.5 hours (150 minutes).

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific or graphing calculator for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on April 1st.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam. Exam is proctored and recorded. You are expected to have your video on at all times.

We acknowledge the following people for their contributions to this year's exam:

*Wesley Antônio Machado Andrade de Aguiar, Erez Abrams, Lucas Pinheiro, Sahil Pontula, Joe McCarty, Hagan Hensley, Leo Yao, and Andrew Liu.*

## 2 Short Questions - 35 points

1. **(10 points)** White Dwarfs are stars in their last stage of life that are prevented from collapsing only by the electron degeneracy pressure. This pressure is an outward one exerted by the electrons inside the star, which are fermions subject to the Pauli exclusion principle. We can find its value by the following formula, which is derived from the theory of fermion gases:

$$p_{\text{electron}} = \frac{2}{3}u = \frac{\hbar^2}{5m_e}(3\pi^2)^{2/3}n^{5/3}$$

where $n$ is the number density of electrons in the star. This pressure balances the inward gravitational pressure, which is given by

$$p_{\text{grav}} = -\frac{\Omega}{3V}, \;\; \Omega = -\frac{3GM^2}{5R}$$

where $\Omega$ is the value of the total potential energy of the star.

   (a) **(8 points)** If the star contains nuclei with atomic number $Z$ and mass number $A$, what is the density value of the white dwarf in function of its total mass $M$, $A$, $Z$, and other fundamental constants?

   (b) **(2 points)** Find what is the value of $k$ in the relation $M \propto V^k$, where $V$ is the volume of the star.

2. **(10 points)** The Large Magellanic Cloud (LMC) is a galaxy with a redshift of $z = 8.75 \times 10^{-4}$.

   (a) **(4 points)** What is the radial velocity of the LMC with respect to the Solar System? Is is getting closer or farther from the Solar System?

   (b) **(4 points)** Hubble's Law is a well-known method of calculating the distance to a galaxy. Using this approach, calculate the distance between the LMC and the Solar System.

   (c) **(2 points)** Is Hubble's Law a reasonable method to determine the distance to the LMC? Explain your answer.

3. **(15 points)** *Culmination Time*

   (a) **(12 points)** In Lubbock, Texas ($\lambda = 101^\circ\, 53'\,\mathrm{W}$, $\phi = 33^\circ\, 35'\,\mathrm{N}$) on September 22nd, what is the local time of upper culmination of Vega ($\alpha = 18\,\mathrm{hr}\,37\,\mathrm{min}$, $\delta = 38^\circ\, 47'$)? The time zone of Lubbock is CDT, UTC$-5$. Assume that the equation of time is 6 min at the relevant time, in the convention of solar time minus mean time.

   (b) **(3 points)** Name the two primary factors which contribute to the equation of time, and give a brief one-sentence explanation for why each causes solar time to differ from mean time.

## 3 Medium Questions - 65 points

1. **(20 points)** *Hyperfine splitting*

   The 21cm spectral line of hydrogen is a result of the interaction between the electron's and proton's quantum mechanical spin (known as hyperfine splitting). The spins can be either aligned or anti-aligned. When a hydrogen atom decays from the higher energy state to the lower energy state, a photon is emitted with energy equal to the energy difference in these two states.

   The intrinsic magnetic moment of the electron $\mu_e$ is approximately equal to the Bohr magneton $\mu_B = \frac{e\hbar}{2m_e}$, and the intrinsic magnetic moment of the proton $\mu_p$ is roughly $2.8\frac{e\hbar}{2m_p}$. Electrons and protons (and any particles with spin $\frac{1}{2}$) have permanent magnetic dipole moments $\vec{m}$ with magnitude equal to the intrinsic magnetic moment.

   (a) **(5 points)** Consider two classical magnetic dipoles $\vec{m}_1, \vec{m}_2$ separated by a distance $\vec{r} = r\hat{r}$. The magnetic field from a perfect magnetic dipole $\vec{m}$ is

   $$\vec{B} = \frac{\mu_0}{4\pi}\frac{3\hat{r}(\vec{m}\cdot\hat{r}) - \vec{m}}{r^3}$$

   The potential energy of a magnetic dipole in an external magnetic field $\vec{B}$ (e.g. the field created by the other dipole) is $U = -\vec{m}\cdot\vec{B}$.

   Using these two formulas together, what is the energy $U_{\text{int}}$ of the interaction between two magnetic dipoles $\vec{m}_1, \vec{m}_2$ separated by a distance $\vec{r} = r\hat{r}$?

   (b) **(6 points)** In our case, we can assume that $r$ is the Bohr radius $a_0$, and both $\vec{m}_1$ and $\vec{m}_2$ are oriented perpendicularly to $\vec{r}$. What is the interaction energy $U_{\text{int}}$ between the magnetic moments of the proton and the electron? Write your answer in terms of $\vec{m}_p \cdot \vec{m}_e$.

   (c) **(2 points)** Which state is the lower energy state: the state where the spins are aligned, or the state where the spins are anti-aligned?

   (d) **(7 points)** The energy of the photon emitted by the transition between these two states is equal to the energy difference in the two states. What value would you predict for the wavelength of the 21cm spectral line? (You should get the right answer to within an order of magnitude, but note that we are treating a quantum mechanical system classically and so there will be large errors.)

2. **(20 points)** Astrophysics studies both the smallest and largest scales of physics. The latter is one of the main focuses of cosmology. Here, we look into the former - the role of quantum mechanics in the astrophysics of stars. For this problem, assume that the Sun's core has a proton number density of $n_c \approx 6 \times 10^{31}$ m$^{-3}$ and temperature $T_c \approx 15$ million K.

   (a) **(2 points)** Suppose that two hydrogen nuclei (protons) are flying towards each other in equal and opposite directions, with an impact parameter (distance of closest approach) of $d \approx 1$ fm. If all of this initial kinetic energy came from the average thermal energy of an ideal gas, what would be the requisite temperature $T_{\text{classical}}$ for "fusion" to occur?

   In quantum mechanics, particles are described by wavefunctions $\psi$, whose squared norms $|\psi|^2$ govern the probability of finding the particle in a particular state (e.g., having a certain position, momentum, or energy). Particles are said to behave as waves with a wavelength given by the de Broglie wavelength $\lambda = h/p$, where $p$ is the particle's momentum and $h$ is Planck's constant. It's also convenient to define $p = \hbar k$, where $k$ is the wavevector and $\hbar = h/(2\pi)$.

   (b) **(2 points)** Evaluate the de Broglie wavelength $\lambda_{\text{dB}}$ in fm (1 fm = $10^{-15}$ m) for a proton with the average thermal energy of an ideal gas at the temperature of the Sun's core, $T_{\text{c}}$.

In Figure 1, we see that it's classically impossible for a nucleus to make it through the potential barrier with $U_0 > E$ (potential energy larger than the total energy). However, quantum mechanics gives a nonzero probability to "tunnel" through the barrier. To solve for $\psi$ in the regions before, inside, and after the barrier, we can use an ansatz $\psi(x) = Ae^{ik(x)x}$, where $A$ is a normalization constant and $k(x) = \sqrt{\frac{2m}{\hbar^2}(E - V(x))}$.

(c) **(2 points)** In which regions of Figure 1 is $k$ purely real? Purely imaginary?

(d) **(2 points)** Modeling the Sun's main fusion reaction as a single process where two protons and two neutrons combine to form a He-4 nucleus, calculate how much energy is released in a single fusion reaction.

(e) **(6 points)** Not every proton-proton collision results in fusion. Calculate the probability $P_{\text{fusion}}$ of fusion per proton necessary to sustain the Sun's current luminosity. You may assume 1% of the Sun's mass is comprised of protons available for fusion. For this part, treat the fusion reaction as a collision between two protons (i.e. ignore the neutrons). Assume the cross section for collision is given by $\pi\lambda_{\text{dB}}^2$ where $\lambda_{\text{dB}}$ is the de Broglie wavelength corresponding to being in thermal equilibrium at temperature $T_c$.

(f) **(6 points)** Consider one proton (moving) in the potential of the other (at rest) in the simplified 1D model of Figure 1. Suppose that $U_0$ is given by the Coulomb repulsion energy of the two protons at a distance $d \approx 1$ fm and that $b$ is the approximate width of the barrier. Here, $b$ is the impact parameter for a proton repelled by the Coulomb force with initial kinetic energy (outside the barrier) equal to the average thermal energy $E$. Assuming the potential energy $V(x) = 0$ outside the barrier, find the temperature $T_{\text{quantum}}$ such that the probability that protons can tunnel through the barrier and fuse is $P_{\text{fusion}}$.

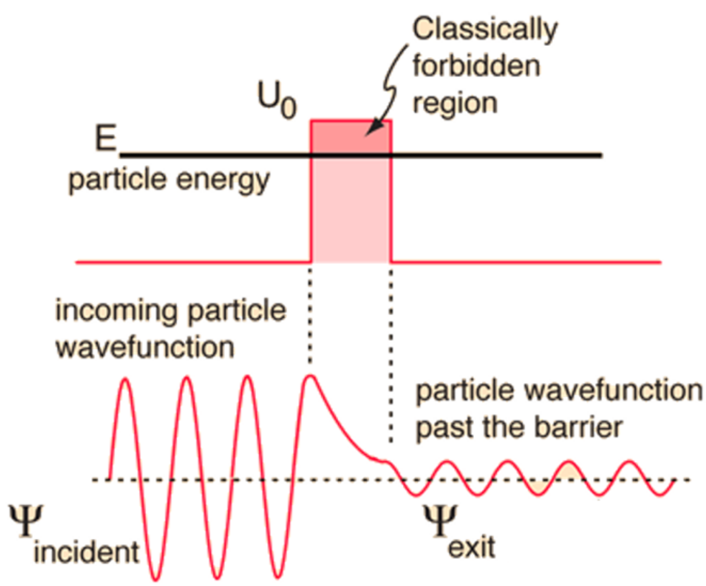



Figure 1: Quantum tunneling.

3. **(25 points)** *Into the Wilds*

You are an astronaut exploring uncharted parts of space in your trusty spaceship, when suddenly you fall into a magical portal to another universe!

Looking for a way back home, you fly to the nearest solar system. Just like normal solar systems, it is home to several planets in orbit around a star - except with one minor difference: it's miniature! The star is only 4 kilometers in diameter, and the whole solar system could fit within a small country back home on Earth.

You know your astrophysics well enough to know that this should be impossible! The only explanation is that some law of physics works differently in this universe. Since you're still alive, chemistry must be unchanged, so quantum mechanics and electromagnetism must work the same as you're used to. Therefore, the only thing that can be different is gravity.

You decide to do some experiments around the solar system to figure out exactly what is different about gravity here.

(a) **(10 points)** First you measure the orbital periods $t_i$ and orbital semi-major axes $a_i$ of the planets in the solar system. You name the planets $P_1$ through $P_5$, because you are incredibly unoriginal. Your data is as follows:

| Planet | $a$ (km) | $t$ (minutes) |
|---|---|---|
| $P_1$ | 5.1 | 1:51 |
| $P_2$ | 9.0 | 4:10 |
| $P_3$ | 12.1 | 6:38 |
| $P_4$ | 17.0 | 11:02 |
| $P_5$ | 20.3 | 14:46 |

Based on this data, do you think that gravity here follows an inverse square law? Justify your answer with calculations and/or drawing an appropriate plot. (You needn't be too formal with statistics here, any well-reasoned argument will get full credit.)

(b) **(3 points)** If the mass of the star is $M$ and the gravitational constant in this universe is $G'$, then what is $G'M$?

This is a good start! Unfortunately, without a known mass, you can't determine the gravitational constant $G'$. You decide to give the planets a closer look to see what else you can find out. You'll start from the outermost and work your way in.

(c) **(5 points)** After narrowly escaping being eaten by $P_5$'s local fauna, you penetrate the stormy atmosphere of $P_4$ to find that it's made almost entirely of water ($\rho = 997 \frac{\text{kg}}{\text{m}^3}$) with a radius of 500 m. While standing in your ship floating on the water's surface, you experience a heavy gravitational pull - 2 times what you're used to on Earth.

What is the gravitational constant in this universe?

(d) **(2 points)** What is the mass of this solar system's star?

(e) **(5 points)** You notice that this star looks yellowish-orange, suggesting its surface temperature is a little lower than that of the Sun. Taking a spectrum, you measure that this star's spectral radiance peaks at a wavelength of 916 nm.

The lifetime of a star can be roughly estimated based on its luminosity and the mass of fuel it has available to burn*. Use a simple scaling argument to roughly estimate the lifetime of this solar system's sun. Explain your reasoning.

(Hint: use our Sun's lifetime of $10^{10}$ years as a reference point.)

*A more careful calculation shows that the temperature at the core of the star wouldn't actually be enough to ignite fusion, so nuclear physics must work differently here in order for fusion to be possible. Let's ignore that for the sake of this question, though.

# 4 Long Questions - 140 points

1. **(45 points)** *The Sundial II*

After gaining an understanding of a sundial, and how the path of the shadow is a straight line on the equinoxes running from due West to due East, Leo now wants to create his own.

At 6am on March 20th, as the Sun is rising, Leo, who is at $(40°\mathrm{N}, 75°\mathrm{W})$, plants a (straight) stick vertically on the ground. At that moment, he marks out a (finite) line on the ground in the direction of the shadow of the stick at that moment, labeling it with the current time. Every hour afterwards, on the hour, he marks out a new line in the current direction of the shadow.

For the following two parts, either show a calculation or explain your answer for credit. Calculating specific values is not necessary, but may help with later parts.

**(10 points)**

(a) Is the angle between the lines corresponding to 12pm and 1pm greater than, equal to, or less than $15°$?

(b) Is the angle between the lines corresponding to 5pm and 6pm greater than, equal to, or less than $15°$?

At 6pm, after drawing his last line, the sun sets. Leo starts cleaning up his setup, taking down the stick. To check his calculations against experiment, Leo measures the angles between pairs of lines on the ground. Using a very precise protractor, he gets the following values (rolling sums are also provided):

| Lines | Angle (degrees) | From 6am (degrees) |
|---|---|---|
| 6am-7am | 9.00570 | 9.00570 |
| 7am-8am | 9.48468 | 18.49038 |
| 8am-9am | 10.90026 | 29.39064 |
| 9am-10am | 13.56767 | 42.95832 |
| 10am-11am | 17.77405 | 60.73237 |
| 11am-12pm | 22.66646 | 83.39883 |
| 12pm-1pm | 24.81144 | 108.21027 |
| 1pm-2pm | 21.90336 | 130.11363 |
| 2pm-3pm | 16.95555 | 147.06918 |
| 3pm-4pm | 13.00713 | 160.07630 |
| 4pm-5pm | 10.58101 | 170.65731 |
| 5pm-6pm | 9.34269 | 180.00000 |

Notably, he finds that they deviate from the values he expects! After frantically checking his calculations and measurements and finding no discrepancies, Leo suspects that the stick might have been slightly tilted. Unfortunately, he took down the stick already, and so can no longer measure it directly.

To help Leo out, we'd like to reconstruct the parameters of the stick tilt. As you work through the following parts, keep in mind this overall goal; it may help to think through all the steps first before proceeding. Ignore atmospheric refraction and the equation of time.

**(35 points)**

(c) To start, define a 2-D coordinate system for the ground, where the $x$-axis points due North and the $y$-axis points due East. Choose and clearly indicate an appropriate origin.

(d) Choose a specific time during the day when the Sun is above the horizon. In your coordinate system, compute the coordinates of the tip of the shadow. Clearly indicate the chosen time and any assumptions made, and justify assumptions if they are not general.

(e) From these coordinates and other information, find a condition on the location of the other end of the stick. Provide your answer as a linear equation in the form $y = mx + b$ for some $m, b$.

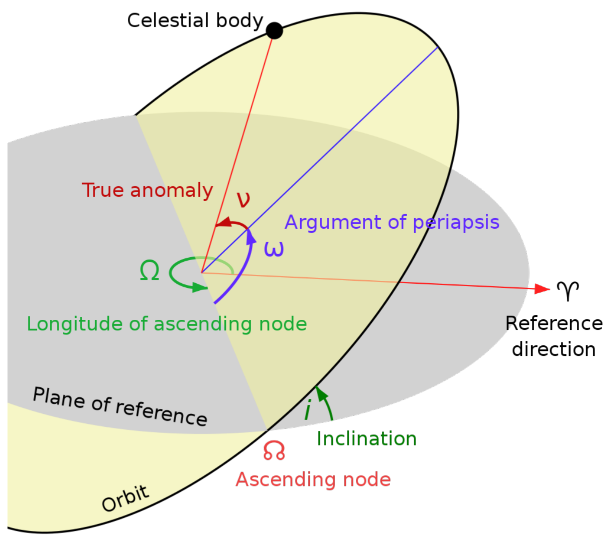
(f) Determine more conditions as necessary, and solve to find the coordinates of the other end of the stick.

(g) Determine the angle, from the vertical, that the stick was tilted, and in what direction.

Depending on method chosen, points on subparts may vary. **Due to small values involved, it is recommended to carry calculations to at least 4 decimal places. Answers to the final part will be integers. No credit given for guesses without justification.**

2. **(40 points)** A new interstellar comet is about to enter the solar system, and Erez is really excited to calculate some Keplerian elements on its approach and figure out where it's going! The comet, dubbed Scratcher Crab by the astronomical community (for no particular reason), is approaching the solar system with a velocity of 29.8km/s from the direction of Altair, which has equatorial coordinates $\alpha = 19^h 51^m, \delta = 8°51'$. If the comet were to continue traveling in a straight line with its current velocity, its closest approach to the sun would be at a distance of 1au, and on March 21st, the comet would briefly be in superior conjunction.

For reference, here is a diagram of orbital elements. Note that the longitude of the ascending node is measured from Aries and the argument of the periapsis is measured from the ascending node:




(a) **(6 points)** What are the eccentricity and semi-major axis of the orbit?

(b) **(6 points)** What are the inclination of the orbit and the longitude of the ascending node, both measured with reference to the equatorial plane (not the ecliptic)?

(c) **(6 points)** What is the angular distance between the incoming and outgoing trajectories of the comet?

(d) **(6 points)** What is the argument of the periapsis?

(e) **(6 points)** What are the exit coordinates of the comet (in equatorial coordinates)?

(f) **(10 points)** Erez wants to check his calculations by making some observations when the comet crosses the plane of the ecliptic. At this time, what is the proper motion of the comet in the sun's frame of reference (not the Earth's)? Give your answer in its equatorial components $\mu_\alpha, \mu_\delta$.

3. **(55 points)** *Finding Friends*

In the future, Earth is perfectly spherical and completely covered with water. Sailor Andrew is on a boat initially positioned at the equator in such that a way that $\gamma$, the vernal equinox, is directly overhead. He observes a star $\chi$ with right ascension and declination $(\alpha, \delta) = (15°, 10°)$. He wants to chase $\chi$, and he needs your help.

**You may assume the following numerical values for each constant:**

- $R_\oplus$ is Earth's radius (same as the current day value).
- $\omega = 2\pi/(24 \text{ hrs})$ is Earth's angular frequency.
- His boat is very fast; its maximum speed is $|v| = \omega R_\oplus/4$.
- The vernal equinox $\gamma$ has celestial coordinates $(0°, 0°)$.
- $(\alpha, \delta) = (15°, 10°)$.

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(a) **(2 points)** Calculate the angular distance between $\gamma$ and $\chi$. Express your answer in degrees.

In his starting position, $\chi$ is somewhere above his horizon. As he sails to different parts of the planet and time passes, $\chi$ moves around his night sky. He wants sail his boat in order to get $\chi$ as close to his zenith as possible. To make sure he is as efficient as possible, he wants to make a few calculations first.

(b) **(2 points)** If he does not move his boat from his starting position, calculate the minimum zenith distance that $\chi$ reaches. Express your answer in degrees.

Note: zenith distance is defined as the angular distance to the zenith.

Now, he considers what would happen if he sails straight north at all times.

(c) **(3 points)** Starting from his intial position when $\gamma$ is overhead, he directs his boat straight north and starts sailing with speed $|v|$. Does $\chi$ first cross his north-south meridian or east-west meridian? Briefly explain why this is the case.

(d) **(3 points)** Calculate the zenith distance of $\chi$ when it first crosses his north-south meridian. Express your answer in degrees.

(e) **(15 points)** Calculate the zenith distance of $\chi$ when it first crosses his east-west meridian. Express your answer in degrees.

Let $X$ be the point in his trajectory (in the celestial sphere) at which $\chi$ reaches its minimum zenith distance, and $N$ be the north celestial pole. Note that $X$ is a point in the celestial sphere, and not Earth.

(f) **(6 points)** Argue that $\angle N\gamma X + \angle \chi X N \approx \pi/2$, where both angles are spherical angles. Then, calculate $\angle \chi X N$. Express your answer in degrees.

Note: For this part only, you may assume that the declination of $X$ is small, hence the approximation.

(g) **(15 points)** Calculate the minimum zenith distance that $\chi$ reaches. Express your answer in degrees. For this problem:

- You may no longer assume that the declination of $X$ is small. However, you may use the value of $\angle \chi X N$ derived in part (f).
- You are given that the right ascension of $X$ is within $(\alpha \pm 2°)$.
- **His trajectory in the celestial sphere is not necessarily a great circle**. Be careful when computing angles and lengths!

He is not satisfied with these results. He feels there should be an optimal way to get $\chi$ directly overhead, as fast as possible. Help him figure out the fastest way to get $\chi$ to his zenith.

(h) **(2 points)** Given that he is allowed to maneuver his boat however he wants throughout the duration of his journey, and that he starts from his initial position with $\gamma$ overhead, show that $\chi$ cannot be at his zenith the first time it culminates, i.e., crosses his north-south meridian.

(i) **(7 points)** Let $t$ be the fastest time he can get $\chi$ to his zenith. Compute the quantity $(t - 20 \text{ hrs})$, in seconds.