2021 National Astronomy Competition

1 Instructions (Please Read Carefully)

The top 5 eligible scorers on the NAC will be invited to represent USA at the next IOAA. In order to qualify for the national team, you must be a high school student with US citizenship or permanent residency.

This exam consists of 3 parts: Short Questions, Medium Questions and Long Questions.

The maximum number of points is 180.

The test must be completed within 2.5 hours (150 minutes).

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific or graphing calculator for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on March 20th.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam. Exam is proctored and recorded. You are expected to have your video on at all times.

After reading the instructions, please make sure to sign, affirming that:

1. All work on this exam is mine.
2. I did not receive any external aids besides the materials provided.
3. I am not allowed to discuss the test with others throughout the period of this examination.
4. Failure to follow these rules will lead to disqualification from the exam.

Signed: ______________________________

2 Short Questions

1. (**5 points**) Jupiter emits more energy to space than it receives from the Sun. The internal heat flux of Jupiter can be quantified by the "intrinsic" temperature of the planet $T_{\rm int}$. The effective temperature $T_{\rm eff}$ of a planet is related to its intrinsic temperature and equilibrium temperature $T_{\rm eq}$ by $T_{\rm eff}^4 = T_{\rm eq}^4 + T_{\rm int}^4$. Given that Jupiter's albedo is 0.5, its emissivity is 1, its average separation from the Sun is 5.2 AU, and its effective temperature is 134 K, estimate its intrinsic temperature in Kelvin. You may use the Sun's surface temperature equal to 5777 K.

> **Solution:** Solve for equilibrium temperature, by equating solar flux falling on Jupiter to a emission of a black-body at temperature T_{eq}
>
> $$T_{\rm eq} = \left(\frac{R_{\rm Sun}}{a}\right)^{1/2} \left[\frac{(1-A)}{4\epsilon}\right]^{1/4} T_{\rm Sun}, \tag{1}$$
>
> Here, A is the albedo, ϵ is the emissivity, a is Jupiter-Sun separation.
>
> $$T_{\rm eq} = \left(\frac{6.96 \times 10^8 \text{ m}}{5.2 \times 1.5 \times 10^{11} \text{ m}}\right)^{1/2} \left[\frac{(1-0.5)}{4}\right]^{1/4} 5777 \text{ K} = 102 \text{ K}. \tag{2}$$
>
> Plug in to solve for intrinsic temperature
>
> $$T_{\rm int} = \left(T_{\rm eff}^4 - T_{\rm eq}^4\right)^{1/4} = \left[(134 \text{ K})^4 - (102 \text{ K})^4\right]^{1/4} = 121 \text{ K}. \tag{3}$$
>
> Note: Accept slightly higher values if the student uses the Solar constant rather than Stefan-Boltzmann law to estimate the Sun's emitted flux.

2. (**5 points**) The convection zone of the sun is the major region of the solar interior that is closest to the surface. It is characterized by convection currents that quickly carry heat to the surface. As a pocket of gas rises, it expands and becomes less and less dense. For it to continue to rise, the temperature gradient in the sun must be steeper than the adiabatic gradient, which is the temperature that the gas would have if it were allowed to expand without any heat input.

   In the sun, the adiabatic gradient satisfies $T \propto p^{0.4}$, where T is the temperature and p is the pressure at any given point.

   The bottom of the convection zone is about 200,000 kilometers beneath the surface of the sun, and has a temperature of about 2×10^6 K and a density of about $200\,\text{kg/m}^3$. Estimate an upper bound for the temperature of the convection zone where the density is $1.2\,\text{kg/m}^3$ (the density of air). You may assume the ideal gas law holds in the convective zone.

> **Solution:** Since the temperature gradient is steeper than the adiabatic gradient, we can get an upper bound for the temperature by assuming the temperature gradient follows the adiabatic gradient exactly.
>
> For a fixed amount of gas molecules, we have that $p \propto \frac{T}{V}$ by the ideal gas law. Also, the adiabatic gradient has $T \propto p^{0.4}$, so we get
>
> $$T \propto \left(\frac{T}{V}\right)^{0.4} = \frac{T^{0.4}}{V^{0.4}}.$$
>
> Using the fact that V is inversely proportional to the density ρ, we have $T \propto T^{0.4}\rho^{0.4}$. Dividing both sides by $T^{0.4}$, we get $T^{0.6} \propto \rho^{0.4}$, so
>
> $$T \propto \rho^{2/3}.$$

Plugging in the numbers we are given, we have that the temperature when the density is equal to that of air is

$$(2 \times 10^6\,\mathrm{K}) \left(\frac{1.2\,\mathrm{kg/m^3}}{200\,\mathrm{kg/m^3}}\right)^{2/3} = \boxed{66000\,\mathrm{K}}.$$

3. (**5 points**) Galaxies are very hard to spot, even those that are nearest to us. For instance, Andromeda, despite having an apparent magnitude of 3.44, appears very "dim" in the sky. This is because its light is very spread out, since its solid angle in the sky is so large (around 3 times that of the Sun!).

   Hence, it is often useful to use the surface magnitude of a galaxy, defined as the magnitude that a certain solid angle of that galaxy has. It is usually measured in $mag/arcmin^2$.

   Show that, in a non expanding universe, the surface magnitude is independent of the distance to the galaxy.

Solution: Let us begin with

$$m_{unit} - m_{total} = -2.5 \log \left(\frac{F_{unit}}{F_{total}}\right)$$

$$m_{unit} - m_{total} = -2.5 \log \left(\frac{\Omega_{unit}}{\Omega_{total}}\right)$$

$$m_{unit} = -2.5 \log(\Omega_{unit}) + 2.5 \log(\Omega_{total}) + m_{total}$$

Where m_{unit} is the surface magnitude, i.e, the magnitude of 1 unit of solid angle; m_{total} is the actual magnitude of the galaxy; F_{unit} is the flux from a unit solid angle; F_{total} is the total flux from the galaxy; Ω_{unit} is the unit solid angle; and Ω_{total} is the total solid angle of the galaxy.

We must show that m_{unit} does not depend on the distance d.

For this, we must proceed as follows:

$$2.5 \log(\Omega_{total}) = 2.5 \log \left(\frac{A}{d^2}\right)$$

$$2.5 \log(\Omega_{total}) = 2.5 \log(A) - 5 \log(d)$$

where A is the physical area of the galaxy and d is the distance to it.

And also, with the distance modulus equation, we can get:

$$m_{total} = M_{total} + 5 \log(d) - 5$$

where M_{total} is the absolute magnitude of the galaxy.

Putting everything together, we have:

$$m_{unit} = -2.5 \log(\Omega_{unit}) + 2.5 \log(A) + M_{total} - 5$$

which is independent of distance, as desired.

4. (**5 points**) An Earth satellite has the following position ($\vec{r}$) and velocity ($\vec{v}$) vectors at a given instant:

$$\vec{r} = 7000\hat{\imath} + 9000\hat{\jmath}(\text{km})$$

$$\vec{v} = -2\hat{\imath} + 5\hat{\jmath}\ (\text{km/s})$$

Calculate the eccentricity of the satellite orbit. Hint: The eccentricity of the orbit is related to total energy E and angular momentum L as $e = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}$; where M is Earth's mass and m is the mass of the satellite.

Solution: Method 1: For a particle experiencing a force in the form of $\vec{F(r)} = -\frac{k}{r^2}\hat{r}$, the Laplace–Runge–Lenz vector is defined as follows:

$$\vec{A} = \vec{P} \times \vec{L} - mk\hat{r}$$

where $\vec{P}$ is the linear momentum and $\vec{L}$ is the angular momentum. The eccentricity can be calculated from the Laplace–Runge–Lenz vector:

$$e = \frac{|A|}{mk}$$

$$e = \frac{|m\vec{v} \times (\vec{r} \times m\vec{v}) - mGMm\hat{r}|}{mGMm}$$

$$e = |\frac{\vec{v} \times (\vec{r} \times \vec{v})}{GM} - \hat{r}|$$

$$e \sim 0.53$$

Method 2: One can also compute the energy ($\frac{E}{m} = \frac{v^2}{2} - \frac{GM}{r}$) and the angular momentum ($\frac{L}{m} = |\vec{r} \times \vec{v}|$) of the orbit and use the following equation to get the eccentricity:

$$e = \sqrt{1 + 2\frac{\frac{E}{m}(\frac{L}{m})^2}{G^2M^2}} \sim 0.53$$

5. (**5 points**) An astronomer who lives in Chicago ($\phi = 41.88^o N$; $\lambda = 87.63^o W$) was very bored during the day of the winter solstice in the Northern hemisphere, so he started thinking about the sunset. The astronomer could not wait to see the sunset on that day. Considering that the true solar time at his location was 2:30 pm, how long did he have to wait to see the sunset? The declination of the sun on winter solstice is $\delta = -23.44^o$.

Solution:

Consider the spherical triangle formed by the Sun, the zenith and the north pole. Let H be the hour-angle, h be the Sun's altitude and δ be the declination of the Sun.

$$\cos(H) = \frac{\sin(h) - \sin(\phi)\sin(\delta)}{\cos(\phi)\cos(\delta)}$$

Since $h = 0^o$ for the sunrise and the sunset:

$$\cos(H) = -\tan(\phi)\tan(\delta)$$

For the winter solstice in the Northern hemisphere, $\delta = -23.44^o$. We get $H = \pm 67.11^o = 4.474\,h$.

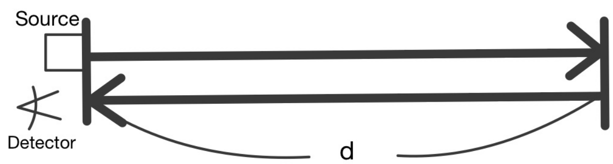
Therefore, the Sun rises 4.474 hours before true solar noon and sets 4.474 hours after true solar noon. Let δt_1 be the time the astronomer has to wait for sunset after $2:30\,\text{pm}$.

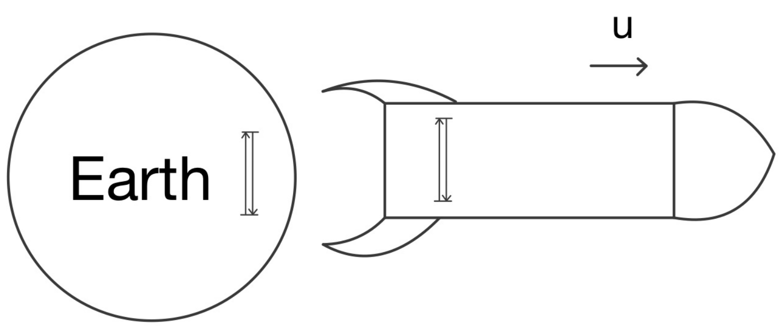
$$\Delta t_1 = 12:00pm + 4.474h - 2:30pm$$
$$\Delta t_1 = 1h\,58min$$

$\therefore$ The astronomer would have to wait 1h 58min to see the sunset.

3 Medium Questions

1. (**20 points**) To measure the time accurately outside the Earth, the engineers build a special clock, with the design as follows: there is a source of light that sends light particles (photons) straight to the reflector that is located at the distance d away from the source. The reflector sends the photons back to their starting point, where there is a detector. This can measure the time accurately, because the speed of light c is constant everywhere. Then a group of engineers built a spaceship with this special clock inside. This spaceship with a clock started to move really fast at the speed u. While the observer in the spaceship reported no issues with the clock inside the spaceship, the observer on the Earth has noticed that the clock is functioning differently in a fast moving spaceship than it is on Earth.





(a) Given that the clock is at rest, what is the total traveling time (Δt_E) of a photon from its source back to the detector?

(b) What is the total distance traveled by a photon d_γ from the source back to the detector on the spaceship moving at the speed u away from the Earth? (Here, we denote that the total traveling time of the photon as Δt_S)

(c) What is the total time Δt_S of a photon as it travels from the source to the detector on the moving spaceship? Answer in terms of d, c, and $\beta = \frac{v}{c}$.

(d) If we relate Δt_E (non-moving frame) to Δt_S (moving frame), as follows: $\Delta t_S = \gamma \Delta t_E$, what does γ equal to? What is significant about the range of γ?

(e) So far, we have only analyzed the motion on the perspective of an observer on the Earth. From the perspective of an observer on the moving spaceship, how do the time on the spaceship $\Delta t_S{}'$ and the time on the Earth $\Delta t_E{}'$ relate to each other?

(f) What can we conclude about the relative passing on time on two different frames that are relatively in motion to one another?

Solution:

(a) $\Delta t_E = \frac{2d}{c}$.

(b) While the photon travels from the source to the reflector, the spaceship moves by $u \cdot \frac{\Delta t_S}{2}$. The distance from the source to the reflector can be then obtained by Pythagorean theorem.

$$\frac{d_S}{2} = \sqrt{d^2 + (u \cdot \frac{\Delta t_S}{2})^2}$$

. Therefore,

$$d_S = 2\sqrt{d^2 + (u \cdot \frac{\Delta t_S}{2})^2}.$$

(c) The speed of light c is constant everywhere. Therefore, the distance traveled by light during Δt_S is $c \cdot \Delta t_S$.

From the last section,

$$2\sqrt{d^2 + (u \cdot \frac{\Delta t_S}{2})^2} = c \cdot \Delta t_S.$$

$$4d^2 + u^2 \Delta t_S{}^2 = \Delta t_S{}^2 \cdot c^2.$$

$$\Delta t_S{}^2 (c^2 - u^2) = 4d^2.$$

$$\Delta t_S = \frac{2d}{\sqrt{c^2 - u^2}} = \frac{2d}{c\sqrt{1 - \frac{v^2}{c^2}}},$$

which equals to

$$\frac{2d}{c} \cdot \frac{1}{\sqrt{1 - (\frac{v}{c})^2}},$$

or

$$\frac{2d}{c} \cdot \frac{1}{\sqrt{1 - \beta^2}}.$$

(d)

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$

Since no other speed can exceed the speed of light c, β is always less than 1. Therefore γ is always greater than 1.

(e)

$$\gamma \Delta t_S{}' = \Delta t_E{}'.$$

Time moves slower on the Earth, because by relativity, the "moving spaceship" observer thinks that the Earth is flying away with a speed u.

(f) Time moves slower in a moving frame when observed from a frame at rest.

2. (**30 points**) You want to send a rocket with an instrument to analyze the atmosphere of Jupiter. In order to get there, you decide to use a Hohmann transfer orbit. $r_E = 1$ AU and $r_J = 5$ AU represent the radii of Earth's and Jupiter's circular orbits around the Sun, respectively. m, M_E, M_J, and M_S represent the masses of your rocket, Earth, Jupiter, and Sun, respectively. Ignore planetary gravitational influences. You may use any other variables you would like if you clearly define them first. Refer to the figures at the end of the question. Show your work for all derivations.

(a) Explain which two (relevant) physical quantities are conserved during this transfer orbit. Write down their statements mathematically.

(b) How long will it take to reach Jupiter?

(c) Halfway through its path to Jupiter, an unrealistic comet passes right next to your rocket and its icy tail freezes your rocket fuel. What is the **maximum** amount of time that you can afford to pass until you need the fuel to be once again unfrozen?

(d) Knowing that this comet will come in the way, your colleague suggests a bi-elliptic transfer orbit instead, with a peak distance of $12r_E$. Write equations describing how long it will now take to reach Jupiter. Will this solution always avoid the comet?

Now that you've compared the orbital times, you want to try and calculate the difference in efficiency.

(e) Derive the δv for each orbital transition in the Hohmann transfer, and sum them to find the total δv.

(f) Derive the δv for each orbital transition in the Bi-elliptic transfer, and sum them to find the total δv.

(g) Factoring in all your previous results, which transfer would you like to use? Why?

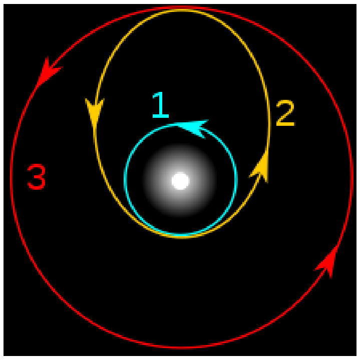


Figure 1: Hohmann Transfer

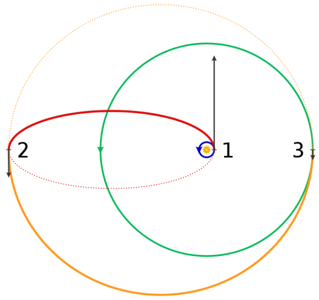


Figure 2: Bi-elliptic Transfer

Solution:

(a) The relevant preserved quantities are energy and angular momentum.

Preservation of angular momentum is given by the following: $L = mv_E r_E = mv_J r_J$ where v_E represents the rocket velocity at the Earth and v_J at Jupiter. More generally, $L = mv_1 r_1 = mv_2 r_2$.

Preservation of energy is given by the following: $E_{TOT} = \frac{mv_E^2}{2} - \frac{GmM_S}{r_E} = \frac{mv_J^2}{2} - \frac{GmM_S}{r_J}$. More generally, $E_{TOT} = \frac{mv_1^2}{2} - \frac{GmM_S}{r_1} = \frac{mv_2^2}{2} - \frac{GmM_S}{r_2}$.

(b) This can be calculated by first specifying the parameters of the rocket's elliptical orbit, and then using Kepler's third law.

Key: use Kepler's third law. The semi-major axis is the average of the minimum and maximum distances from the sun. $a = \frac{r_E + r_J}{2}$. If this is specified in AU, we can use $P^2 = a^3$ where P is in years. Thus, $P = \left(\frac{r_E + r_J}{2}\right)^{\frac{3}{2}}$. However, we only want to take half of this orbital period, since we are stopping at Jupiter rather than coming all the way back to Earth's orbital radius. Thus, we simply take half of this value:

$$T = \frac{1}{2}\left(\frac{r_E + r_J}{2}\right)^{\frac{3}{2}}$$

.

(c) You can take an unlimited amount of time to wait for the fuel to unfreeze. This is because we are ignoring gravitational effects from planets, and thus the rocket will constantly stay in the elliptical orbit with Jupiter's orbit at apoapsis and Earth's orbit at periapsis. Whenever the fuel unfreezes, we would wait from that point on for the rocket to reach apoapsis before firing engines for δv_2.

(d) Remember that a bi-elliptic transfer has one free parameter: the joint apoapsis X AU from the sun, or the max distance from the sun the rocket will reach. Thus, we immediately know this solution will not always avoid the comet. If we set the joint apoapsis to be just barely greater than Jupiter's orbit, the first elliptical orbit will be almost identical to the Hohmann transfer orbit.

Using similar logic from part b, the time to reach Jupiter would be $T = \frac{1}{2}a_1^{\frac{3}{2}} + \frac{1}{2}a_2^{\frac{3}{2}}$.

$$a_1 = \frac{12r_E + r_E}{2} = 6.5r_E$$

$$a_2 = \frac{12r_E + r_J}{2} = 6r_E + \frac{r_J}{2}$$

(e) Key: use the vis-viva equation.

Vis-viva equation:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Circular orbit 1 radius: r_E

Semi-major axis of elliptical orbit:

$$a = \frac{r_E + r_J}{2}$$

Circular orbit 2 radius: r_J

Thus, Velocity at Earth in circular orbit =

$$\sqrt{\frac{GM}{r_E}}$$

Velocity at Earth in elliptical orbit =

$$\sqrt{GM\left(\frac{2}{r_E} - \frac{2}{r_E + r_J}\right)}$$

Velocity at Jupiter in elliptical orbit =

$$\sqrt{GM\left(\frac{2}{r_J} - \frac{2}{r_E + r_J}\right)}$$

Velocity at Jupiter in circular orbit =

$$\sqrt{\frac{GM}{r_J}}$$

Now, to find the δv at each stage, compute the differences in velocities between each interface between the elliptical and circular orbits.

$$\delta v_1 = \sqrt{GM\left(\frac{2}{r_E} - \frac{2}{r_E + r_J}\right)} - \sqrt{\frac{GM}{r_E}}$$

$$\delta v_2 = \sqrt{\frac{GM}{r_J}} - \sqrt{GM\left(\frac{2}{r_J} - \frac{2}{r_E + r_J}\right)}$$

$$\delta v_{tot} = \sqrt{GM}\left(\sqrt{\frac{2}{r_E} - \frac{2}{r_E + r_J}} - \sqrt{\frac{1}{r_E}} + \sqrt{\frac{1}{r_J}} - \sqrt{\frac{2}{r_J} - \frac{2}{r_E + r_J}}\right)$$

(f) Key: use the vis-viva equation.

Vis-viva equation:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Semi-major axis of ellipse 1:

$$a_1 = \frac{r_E + X}{2}$$

Semi-major axis of ellipse 2:

$$a_2 = \frac{r_J + X}{2}$$

Velocity at Earth in Earth's orbit:

$$\sqrt{GM}\sqrt{\frac{1}{r_E}}$$

Velocity at Earth in elliptical orbit 1:

$$\sqrt{GM}\sqrt{\frac{2}{r_E} - \frac{1}{a_1}}$$

Velocity at joint apoapsis in elliptical orbit 1:

$$\sqrt{GM}\sqrt{\frac{2}{X} - \frac{1}{a_1}}$$

Velocity at joint apoapsis in elliptical orbit 2:

$$\sqrt{GM}\sqrt{\frac{2}{X} - \frac{1}{a_2}}$$

Velocity at Jupiter in elliptical orbit 2:

$$\sqrt{GM}\sqrt{\frac{2}{r_J} - \frac{1}{a_2}}$$

Velocity at Jupiter in Jupiter's orbit:

$$\sqrt{GM}\sqrt{\frac{1}{r_J}}$$

Using the above equations, we can write all the δv's.

Speed boost to enter elliptical orbit 1:

$$\delta v_1 = \sqrt{GM}\left(\sqrt{\frac{2}{r_E} - \frac{1}{a_1}} - \sqrt{\frac{1}{r_E}}\right)$$

Speed boost to enter elliptical orbit 2:

$$\delta v_2 = \sqrt{GM}\left(\sqrt{\frac{2}{X} - \frac{1}{a_2}} - \sqrt{\frac{2}{X} - \frac{1}{a_1}}\right)$$

Speed reduction to enter Jupiter's orbit:

$$\delta v_3 = \sqrt{GM}\left(\sqrt{\frac{2}{r_J} - \frac{1}{a_2}} - \sqrt{\frac{1}{r_J}}\right)$$

Total δv, replacing $X = 12r_E$:

$$\delta v_{tot} = \sqrt{GM}\left(\sqrt{\frac{2}{r_E} - \frac{1}{a_1}} - \sqrt{\frac{1}{r_E}} + \sqrt{\frac{1}{6r_E} - \frac{1}{a_2}} - \sqrt{\frac{1}{6r_E} - \frac{1}{a_1}} + \sqrt{\frac{2}{r_J} - \frac{1}{a_2}} - \sqrt{\frac{1}{r_J}}\right)$$

(g) This could be algebraically solved. Much more simply, plug in $r_E = 1$ AU and $r_J = 5$ AU to the time and velocity equations. The Hohmann transfer is both faster and more efficient in terms of energy use (δv).

3. (**30 points**) A space station of mass m is orbiting a planet of mass M_0 on a circular orbit of radius r. At a certain moment, a satellite of mass m is launched from the space station with a relative velocity $\overrightarrow{w}$ oriented towards the center of the planet. Assume that $w < \sqrt{\frac{GM_0}{r}}$.

   (a) Justify the shape of the satellite's orbit after launching and, for the satellite-planet system, determine the following quantities:

   (1) Satellite's velocity relative to the planet, immediately after launch, v

   (2) Total angular momentum of the satellite-planet system, $L_{P,Sat}$

   (3) Satellite's orbit semi-major and semi-minor axes, a_{Sat} and b_{Sat}

   (4) Satellite's orbit eccentricity, ϵ_{Sat}

   (5) Apogee and perigee distances, $r_{max,Sat}$ and $r_{min,Sat}$

   (6) Satellite's minimum velocity, $v_{min,Sat}$ and maximum velocity $v_{max,Sat}$ on it's orbit

   (7) Total energy of the satellite-planet system, $E_{Sat,P}$.

   (b) Determine the shape of the space station's orbit relative to the planet, after the satellite was launched.

Solution:

(a) The absolute velocity of the satellite (relatively to the planet) at the moment of launching is:

$$\overrightarrow{v} = \overrightarrow{w} + \overrightarrow{u}$$

,where u is the orbital station's velocity relatively to the planet.

$$v = \sqrt{w^2 + u^2}$$

$$u = \omega_0 r$$

,where ω_0 is the angular velocity of the space station orbiting around the planet, before the satellite was launched.

$$T_0 = \frac{2\pi r}{u} = \frac{2\pi}{\omega_0} = \frac{u}{r}$$

,where T_0 is the orbital period of the space station, before the satellite was launched.

$$\frac{mu^2}{r} = \frac{GmM_0}{r^2}$$

$$u = \sqrt{\frac{GM_0}{r}} \quad \omega_0 = \sqrt{\frac{GM_0}{r^3}}$$

Thus, we get:

$$v = \sqrt{w^2 + \frac{GM_0}{r}} > w$$

So, after the satellite was launched at point Q, it will move on an elliptical orbit, having the planet as a one of the focal points. This happens because the total energy of the satellite - planet system, calculated for the moment when the satellite is injected on its orbit is:

$$E_{Q,S-P} = \frac{mv^2}{2} - \frac{GmM_0}{r} = \frac{m}{2}(w^2 + u^2) - \frac{GmM_0}{r}$$

$$E_{Q,S-P} = \frac{m}{2}(w^2 + \frac{GM_0}{r}) - \frac{GmM_0}{r}$$

$$E_{Q,S-P} = \frac{m}{2}(w^2 - \frac{GM_0}{r})$$

$$w < \sqrt{\frac{GM_0}{r}}; \quad w^2 < \frac{GM_0}{r}; \quad E_{Q,S-P} < 0$$

Having this, we can determine the angular momentum of the space station relatively to the fixed planet, corresponding to point Q (point of injection):

$$\overrightarrow{L_{Q,Sat}} = \overrightarrow{r} \times mv = \overrightarrow{r} \times m(\overrightarrow{w} + \overrightarrow{u})$$

But the angle between $\overrightarrow{r}$ and $\overrightarrow{w}$ is 180°. Thus, $\overrightarrow{r} \times \overrightarrow{w} = 0$. So:

$$\overrightarrow{L_{Q,Sat}} = \overrightarrow{r} \times m\overrightarrow{u}$$

Moreover, the angle between $\overrightarrow{r}$ and $\overrightarrow{u}$ is 90°. Thus, we get:

$$L_{Q,Sat} = mru \quad u = \omega_0 r$$

$$L_{Q,Sat} = mr^2\omega_0 = mr^2\sqrt{\frac{GM_0}{r^3}} = m\sqrt{GrM_0}$$

Evolving on an elliptical orbit around the planet, when the satellite will reach the minimum distance from the planet, its angular momentum will be:

$$\overrightarrow{L_{P,Sat}} = \overrightarrow{r_{min,Sat}} \times m\overrightarrow{v_{max,Sat}}$$

Because the angle between $\overrightarrow{r_{min,Sat}}$ and $\overrightarrow{v_{max,Sat}}$ is 90°, we get:

$$L_{P,Sat} = mr_{min,Sat}v_{max,Sat}$$

Because the total angular momentum is conserved:

$$L_{P,Sat} = L_{Q,Sat} = L_{A,Sat}$$

$$mr_{min,Sat}v_{max,Sat} = m\sqrt{GrM_0} = mr_{max,Sat}v_{min,Sat}$$

Thus, we get:

$$r_{min,Sat}v_{max,Sat} = \sqrt{GrM_0} = r_{max,Sat}v_{min,Sat}$$

$$v_{max,Sat} = \frac{\sqrt{GrM_0}}{r_{min,Sat}}; \quad v^2_{max,Sat} = \frac{GrM_0}{r^2_{min,Sat}}$$

$$v_{min,Sat} = \frac{\sqrt{GrM_0}}{r_{max,Sat}}; \quad v^2_{min,Sat} = \frac{GrM_0}{r^2_{max,Sat}}$$

Also, the total mechanical energy of the satellite - planet system is conserved, thus:

$$E_{Q,S-P} = E_{P,S-P}$$

$$\frac{1}{2}m(w^2 + \frac{GM_0}{r}) - \frac{GmM_0}{r} = \frac{mv^2_{max,Sat}}{2} - \frac{GmM_0}{r_{min,Sat}}$$

$$(w^2 + \frac{GM_0}{r}) - \frac{2GM_0}{r} = v^2_{max,Sat} - \frac{2GM_0}{r_{min,Sat}}$$

$$w^2 - \frac{GM_0}{r} = v^2_{max,Sat} - \frac{2GM_0}{r_{min,Sat}}$$

But:

$$v^2_{max,Sat} = \frac{GrM_0}{r^2_{min,Sat}}$$

Thus:

$$w^2 - \frac{GM_0}{r} = \frac{GrM_0}{r^2_{min,Sat}} - \frac{2GM_0}{r_{min,Sat}}$$

$$(w^2 - \frac{GM_0}{r})r^2_{min,Sat} + 2GM_0 r_{min,Sat} - GrM_0 = 0$$

By solving this, we get:

$$r_{min,Sat} = \frac{r(-GM_0 \pm w\sqrt{GM_0 r})}{w^2 r - GM_0}$$

$$r_{min,Sat} = \frac{r(GM_0 \mp w\sqrt{GM_0 r})}{GM_0 - w^2 r}$$

Because we are searching for the minimum value, we will assign $r_{min,Sat}$ the negative solution. Thus, the positive solution will be the value for $r_{max,Sat}$. Thus:

$$r_{min,Sat} = \frac{r(GM_0 - w\sqrt{GM_0 r})}{GM_0 - w^2 r}$$

$$v_{max,Sat} = \frac{r}{r_{min,Sat}}\sqrt{\frac{GM_0}{r}}$$

Thus, we get;

$$v_{max,Sat} = \frac{GM_0 - w^2 r}{GM_0 - w\sqrt{GrM_0}}\sqrt{\frac{GM_0}{r}}$$

Consequently, for $r_{max,Sat}$:

$$r_{max,Sat} = \frac{r(GM_0 + w\sqrt{GM_0 r})}{GM_0 - w^2 r}$$

$$v_{min,Sat} = \frac{r}{r_{max,Sat}}\sqrt{\frac{GM_0}{r}}$$

Thus:

$$v_{min,Sat} = \frac{GM_0 - w^2 r}{GM_0 + w\sqrt{GrM_0}}\sqrt{\frac{GM_0}{r}}$$

From the properties of the ellipse:

$$r_{min,Sat} + r_{max,Sat} = 2a_{Sat}$$

,where a is the semi-major axis of the ellipse. Thus:

$$\frac{r(GM_0 + w\sqrt{GM_0 r})}{GM_0 - w^2 r} + \frac{r(GM_0 - w\sqrt{GM_0 r})}{GM_0 - w^2 r} = 2a_{Sat}$$

$$a_{Sat} = \frac{GrM_0}{GM_0 - rw^2} > r$$

Using the conservation laws for the total energy and total angular momentum, we prove that:

$$L_{Sat} = mb_{Sat}\sqrt{\frac{GM_0}{a_{Sat}}} = L_{P,Sat} = mv_{max,Sat}r_{min,Sat}$$

$$b_{Sat}\sqrt{\frac{GM_0}{a_{Sat}}} = v_{max,Sat}r_{min,Sat}$$

$$b_{Sat} = v_{max,Sat}r_{min,Sat}\sqrt{\frac{a_{Sat}}{GM_0}} = \sqrt{GrM_0}\cdot\sqrt{\frac{a_{Sat}}{GM_0}}$$

Thus, the semi-minor axis is:

$$b_{Sat} = r\cdot\sqrt{\frac{GM_0}{GM_0 - rw^2}}$$

Now, we can determine the eccentricity of the satellite's orbit:

$$e_{Sat} = \sqrt{1 - \frac{b_{Sat}^2}{a_{Sat}^2}} = \sqrt{1 - \frac{r^2\cdot\frac{GM_0}{GM_0 - rw^2}}{r^2\cdot\left(\frac{GM_0}{GM_0 - rw^2}\right)^2}} = \sqrt{1 - \frac{GM_0 - rw^2}{GM_0}}$$

Thus, we get:

$$e_{Sat} = w\sqrt{\frac{r}{GM_0}}$$

The total energy of the satellite-Earth system is:

$$E_{Sat-E} = -\frac{GmM_0}{2a_{Sat}}$$

,where $a_{Sat} = r\frac{GM_0}{GM_0 - rw^2}$.

$$E_{Sat-E} = -\frac{GmM_0}{2r\frac{GM_0}{GM_0 - rw^2}} = -GmM_0\frac{GM_0 - rw^2}{2rGM_0}$$

$$E_{Sat-E} = -\frac{m(GM_0 - rw^2)}{2r}$$

(b) After the satellite was launched with a radial relative velocity of $\overrightarrow{w}$ towards the planet, the space station also gained a radial velocity $\overrightarrow{W}$ oriented in the opposite direction of $\overrightarrow{w}$ because the total momentum is conserved:

$$\overrightarrow{V} = \overrightarrow{W} + \overrightarrow{u}$$

$$V = \sqrt{W^2 + u^2}$$

Conservation of momentum:

$$M\overrightarrow{W} + m\overrightarrow{w} = 0; \quad W = \frac{m}{M}w; \quad m < M; \quad W < w;$$

$$V = \sqrt{\frac{m^2}{M^2}w^2 + u^2}; \quad V < v;$$

In order to determine the shape of the orbit we need to calculate the total energy of the station-planet system:

$$E_{Sta,P} = \frac{MV^2}{2} - \frac{GMM_0}{r} = \frac{M}{2}(\frac{m^2}{M^2}w^2 + u^2) - \frac{GMM_0}{r}$$

$$E_{Sta,P} = \frac{M}{2}(\frac{m^2}{M^2}w^2 + \frac{GM_0}{r}) - \frac{GMM_0}{r}$$

$$E_{Sta,P} = \frac{1}{2}M\frac{m^2}{M^2}w^2 - \frac{1}{2}\frac{GMM_0}{r}$$

$$E_{Sta,P} = \frac{1}{2}M\frac{m^2}{M^2}\frac{GM_0}{r} - \frac{1}{2}\frac{GMM_0}{r}$$

$$E_{Sta,P} = \frac{1}{2}\frac{GMM_0}{r}(\frac{m^2}{M^2} - 1) < 0$$

Thus, after the satellite is launched, the space station will move on an elliptical orbit around the planet.

4 Long Questions

1. (**40 points**) In the very early universe, everything is in thermodynamic equilibrium and particles are freely created, destroyed, and converted between each other due to the high temperature. In one such process, the reaction converting between neutrons and protons happens at a very high rate. In thermal equilibrium, the relative number density of particle species is given approximately by the Boltzmann factor:

$$n_i \propto \exp\left[-\frac{E_i}{k_B T}\right],$$

where $E_i = m_i c^2$ is the rest energy. Additionally, the temperature during the radiation-dominated early universe is given by $T(t) \approx 10^{10}\,\mathrm{K}\left(\frac{t}{1\,\mathrm{s}}\right)^{-1/2}$, where t is the time since the Big Bang.

(a) (4 points) At a temperature where $k_B T \approx 0.8\,\mathrm{MeV}$, known as the *freeze-out* temperature, the neutrino interactions essentially stop, preventing further conversion between protons and neutrons.

i. (2 points) About how long after the Big Bang did this occur?

ii. (2 points) At the freeze-out temperature, what was the equilibrium ratio of the number density of neutrons to that of protons?

(b) (3 points) Free neutrons are unstable, and decay into protons with a characteristic decay time of $\tau = 886\,\mathrm{s}$ (the time for which the number of neutrons drops to $1/e$ of the original amount). Given that helium nuclei only formed $t_{nuc} = 200\,\mathrm{s}$ after freezing out, what was the ratio of the number density of neutrons to that of protons when the helium nuclei formed?

(c) (7 points) While trace amounts of several small nuclei were formed during Big Bang Nucleosynthesis (BBN), assume that all neutrons go into forming helium-4.

i. (5 points) After the helium nuclei formed, what was the ratio of the number of helium-4 nuclei to the number of hydrogen nuclei?

ii. (2 points) Approximating the mass of helium-4 as 4 times that of H (for this part only), what fraction of baryonic mass in the universe is helium?

If you weren't able to solve part (c), assume reasonable values for the initial mass fractions of hydrogen and helium for future parts.

(d) (2 points) Albert the Astronomer claims that in older galaxies, the mass fraction of hydrogen should gradually be increasing, as neutrons slowly continue to decay into protons. Is his claim correct? If not, explain.

(e) (7 points) Suppose a certain region of a galaxy has a density of $10^{-19}\,\mathrm{kg/m^3}$ and is composed of 70% hydrogen and 30% helium-4 by mass (ignore any heavier elements). Because the region is gravitationally bound, this density doesn't change significantly with the expansion of the universe; approximate it as constant. Assume hydrogen is converted into helium by the fusion reaction:

$$4\,{}^1\mathrm{H}^+ + 2\mathrm{e}^- \rightarrow {}^4\mathrm{He}^{2+} + 2\nu_e,$$

where the electron e^- and electron neutrino ν_e are of negligible mass. ^{4}He has a mass of $m_{He} = 3728.4\,\mathrm{MeV/c^2}$

i. (4 points) Over the entire time since BBN, how much energy does this process release per cubic kiloparsec? Give your answer in joules per cubic kiloparsec.

ii. (3 points) Assuming the age of the universe is 13.8 billion years, calculate the average luminosity density in solar luminosities per cubic kiloparsec.

Let's go back and explore how we arrived at the number $t_{nuc} \approx 200\,\mathrm{s}$, the time at which Big Bang nucleosynthesis began. Let's define t_{nuc} as the time at which half the neutrons fused with protons into deuterium (^{2}H), as deuterium fusion is the first step in BBN. From the Maxwell-Boltzmann equation, the relative abundances of deuterium, protons and neutrons is given by

$$\frac{n_D}{n_p n_n} = 6\left(\frac{m_n k_B T}{\pi \hbar^2}\right)^{-3/2} \exp\left(\frac{B_D}{k_B T}\right),$$

where $B_D = (m_p + m_n - m_D)\,c^2 = 2.22\,\text{MeV}$ is the energy released in a deuterium fusion reaction.

(f) (3 points) The number density of photons is given by $n_\gamma = 0.243\left(\frac{k_B T}{\hbar c}\right)^3$. Find an expression for the number density of protons n_p in terms of the temperature T and the baryon to photon ratio η. You may use your answer to part (b).

(g) (3 points) Find the present-day baryon to photon ratio. The CMB temperature is 2.725 K, and the present-day density parameter for baryonic matter is $\Omega_{b,0} = \frac{\rho_{b,0}}{\rho_{c,0}} = 0.04$. ρ_c is the critical density of the universe, which is the density required for a flat universe; it is given by $\rho_c = \frac{3H^2}{8\pi G}$. Use $H_0 = 70\,\text{km/s/Mpc}$.

(h) (8 points) Assuming the baryon to photon ratio is fixed since the Big Bang:

  i. (5 points) Find an equation involving T_{nuc} (the temperature at time $t = t_{nuc}$) and known constants.

  ii. (1 point) What temperature T_{nuc} does $t_{nuc} = 200\,\text{s}$ correspond to?

  iii. (2 points) Verify that this temperature solves your equation in part (h)i.

(i) (3 points) The baryon to photon η is a remarkably small number. One possibility is that the universe happens to prefer photons significantly over baryons. Another possibility is that a great number of quark-antiquark pairs were created in the early universe via pair production ($\gamma + \gamma \rightleftharpoons q + \bar{q}$), and a slight asymmetry of quarks over antiquarks produced a large number of photons during quark-antiquark annihilation, leaving over a small number of quarks to form into protons and neutrons. Find the quark-antiquark asymmetry

$$\delta_q \equiv \frac{n_q - n_{\bar{q}}}{n_q + n_{\bar{q}}} \ll 1$$

that would yield the baryon to photon ratio found in part (g).

Solution:

(a) i. When $k_B T \approx 0.8\,\text{MeV}$, we have

$$\begin{aligned} T &= \frac{0.8\,\text{MeV}}{k_B} \\ &= \frac{1.28 \cdot 10^{-13}\,\text{J}}{1.381 \cdot 10^{-23}\,\text{J/K}} \\ &= 9.28 \cdot 10^9\,\text{K}. \end{aligned}$$

Now, we have

$$t = \left(\frac{10^{10}}{T}\right)^2 = \boxed{1.16\,\text{s}}.$$

ii. Let n_p be the number density of protons, and n_n the number density of neutrons. Then, we have that the ratio $\frac{n_n}{n_p}$ is

$$\frac{\exp\left[-\frac{m_n c^2}{k_B T}\right]}{\exp\left[-\frac{m_p c^2}{k_B T}\right]} = \exp\left[-\frac{(m_n - m_p)c^2}{k_B T}\right].$$

Plugging in $m_n c^2 - m_p c^2 = 939.6 - 938.3 = 1.3\,\mathrm{MeV}$ and $k_B T = 0.8\,\mathrm{MeV}$, we get $\frac{n_n}{n_p} = \boxed{0.197}$.

(b) For every proton, there is initially 0.197 of a neutron. This decays according to $n_n = n_{n,0}\exp\left(-\frac{t}{\tau}\right)$. We have $0.197\exp\left(-\frac{200\,\mathrm{s}}{886\,\mathrm{s}}\right) = 0.157$, so the new ratio is $\frac{0.157}{1+0.197-0.157} = \boxed{0.151}$ (taking into account that the decayed neutrons turn into protons).

(c) i. Using the answer to part (b), for every 0.151 neutrons, there is 1 proton. Then, they can form $0.151/2 = 0.076$ helium-4 nuclei, and the remaining $1 - 0.151 = 0.849$ protons can form 0.849 hydrogen nuclei.
Thus, the ratio of helium nuclei to hydrogen nuclei is $\frac{0.075}{0.849} = \boxed{0.089}$.

ii. The mass ratio is $4 \cdot 0.089 = 0.356$, which gives a mass percentage of $\frac{0.356}{1+0.356} = \boxed{26.3\%}$

(d) Albert is $\boxed{\text{not correct}}$. Only free neutrons are unstable, and the vast majority of neutrons in the universe are bound up in nuclei, particularly helium-4. Furthermore, fusion in stars actually decreases the fraction of hydrogen, as explored in the following part.

(e) i. Since the electron and electron neutrino are of negligible mass, each reaction releases $4m_p c^2 - m_{He}c^2 = 4 \cdot 938.3\,\mathrm{MeV} - 3728.4\,\mathrm{MeV} = 24.8\,\mathrm{MeV}$. Hydrogen went from a density of $(1-0.263)\cdot 10^{-19}\,\mathrm{kg/m^3} = 7.37\times 10^{-20}\,\mathrm{kg/m^3}$ to $0.70\cdot 10^{-19}\,\mathrm{kg/m^3} = 7\times 10^{-20}\,\mathrm{kg/m^3}$, with a difference of $3.7\times 10^{-21}\,\mathrm{kg/m^3}$. This means that the energy released per cubic kiloparsec is

$$24.8\,\mathrm{MeV}\cdot\frac{3.7\times 10^{-21}\,\mathrm{kg/m^3}}{4\cdot 1.6726\times 10^{-27}\,\mathrm{kg}}\cdot\left(\frac{3.086\times 10^{19}\,\mathrm{m}}{1\,\mathrm{kpc}}\right)^3\cdot\frac{1.6022\times 10^{-13}\,\mathrm{J}}{1\,\mathrm{MeV}} = \boxed{6.5\times 10^{52}\,\mathrm{J/kpc^3}}.$$

ii. In seconds, 13.8 billion years is $13.8\times 10^9\cdot 365.25\cdot 24\cdot 3600\,\mathrm{s} = 4.35\times 10^{17}\,\mathrm{s}$. Thus, the average power per cubic kiloparsec is $\frac{6.5\times 10^{52}\,\mathrm{J/kpc^3}}{4.35\times 10^{17}\,\mathrm{s}} = 1.50\times 10^{35}\,\mathrm{W/kpc^3}$. In solar luminosities, this quantity is $\frac{1.50\times 10^{35}\,\mathrm{W/kpc^3}}{3.85\times 10^{26}\,\mathrm{W}/L_\odot} = \boxed{3.90\times 10^8\,L_\odot/\mathrm{kpc^3}}$.

(f) From part (b), the ratio of neutrons to protons is 0.151. Thus, the proton to baryon ratio is $\frac{1}{1+0.151} = 0.869$, or $n_p = 0.869\,n_b$. By definition, $n_b = n_\gamma\eta = 0.243\eta\left(\frac{k_B T}{\hbar c}\right)^3$. We arrive at the expression $\boxed{n_p = 0.211\eta\left(\frac{k_B T}{\hbar c}\right)^3}$.

(g) Let us first find the present-day number density of photons. Simply plugging in $T = 2.725\,\mathrm{K}$ into the expression given in part (f), we have

$$n_\gamma = 0.243\left(\frac{k_B\cdot 2.725\,\mathrm{K}}{\hbar c}\right)^3 = 4.09\cdot 10^8\,\mathrm{photons/m^3}.$$

To find the present-day number density of baryons, we first need to find the present-day critical density. Using $H_0 = 70\,\mathrm{km/s/Mpc} = 2.3\cdot 10^{-18}\,\mathrm{s^{-1}}$, we have $\rho_{c,0} = \frac{3H_0^2}{8\pi G} = 9.2\cdot 10^{-27}\,\mathrm{kg/m^3}$. Since $\Omega_{b,0} = 0.04$, $\rho_{b,0} = 0.04\cdot 9.2\cdot 10^{-27}\,\mathrm{kg/m^3} = 3.7\cdot 10^{-28}\,\mathrm{kg/m^3}$. Finally, baryonic matter is composed of protons and neutrons; since $m_p \approx m_n$, we can divide by the mass of a proton and find the number density of baryons

$$n_b = \frac{3.7\cdot 10^{-28}\,\mathrm{kg}}{\mathrm{m^3}}\cdot\frac{1\,\mathrm{baryon}}{1.6726\cdot 10^{-27}\,\mathrm{kg}} = 0.22\,\mathrm{baryons/m^3}.$$

The baryon to photon ratio is $\eta = \boxed{5.4\cdot 10^{-10}}$.

(h) i. Earlier, we defined t_{nuc} to be the time at which half the neutrons fused into deuterium, or $\frac{n_D}{n_n} = 1$. Setting $\frac{n_D}{n_n} = 1$ and plugging in our expression for n_p from part (f), we find

$$1 = 0.211\eta \left(\frac{k_B T}{\hbar c}\right)^3 \cdot 6 \left(\frac{m_n k_B T}{\pi \hbar^2}\right)^{-3/2} \exp\left(\frac{B_D}{k_B T}\right)$$

$$\boxed{1 \approx 7\eta \left(\frac{k_B T_{nuc}}{m_n c^2}\right)^{3/2} \exp\left(\frac{B_D}{k_B T_{nuc}}\right)}$$

ii. Using $T(t) \approx 10^{10}\,\mathrm{K}\left(\frac{t}{1\,\mathrm{s}}\right)^{-1/2}$, we get $T(t = 200\,\mathrm{s}) = \boxed{7 \cdot 10^8\ \mathrm{K}}$. The corresponding energy is $k_B T_{nuc} = 0.061\,\mathrm{MeV}$.

iii. Using $k_B T_{nuc} = 0.061\,\mathrm{MeV}$, we have

$$7\eta \left(\frac{k_B T_{nuc}}{m_n c^2}\right)^{3/2} \exp\left(\frac{B_D}{k_B T_{nuc}}\right)$$

$$= 7 \cdot 5.4 \cdot 10^{-10} \left(\frac{0.061\,\mathrm{MeV}}{939.6\,\mathrm{MeV}}\right)^{3/2} \exp\left(\frac{2.22\,\mathrm{MeV}}{0.061\,\mathrm{MeV}}\right) = 13$$

Due to the exponential, this expression is very sensitive to small changes in $k_B T$. Thus for this expression, an answer of 13 is roughly consistent with 1. The exact solution is $k_B T = 0.066$ MeV, which still corresponds to $t_{nuc} = 200$ s to the nearest significant figure.

(i) $2n_{\bar{q}}$ quarks are annihilated, producing $2n_{\bar{q}} \approx n_q + n_{\bar{q}}$ photons. $n_q - n_{\bar{q}}$ quarks are left over to form $(n_q - n_{\bar{q}})/3$ baryons. The resulting baryon to photon ratio is thus

$$\eta = \frac{(n_q - n_{\bar{q}})/3}{n_q + n_{\bar{q}}} = \frac{1}{3}\delta_q.$$

Using $\eta = 5.4 \cdot 10^{-10}$, we find a quark-antiquark asymmetry of $\boxed{\delta_q = 1.6 \cdot 10^{-9}}$; there was one extra quark in 800 million quark-antiquark pairs.

2. (**35 points**) In 2020, during the day of the winter solstice for the Northern hemisphere, Jupiter and Saturn were at their minimum angular separation (approximately 6.11') during the Great Conjunction.

(a) (6 points) Consider a system with three planets in circular, concentric, and coplanar orbits around a star. Suppose that the three planets and the star are initially aligned. Will they necessarily align again after this moment? Prove your answer with quantitative arguments. Assume that the sidereal periods of all planets are rational numbers in terms of some unit period.

(b) (4 points) Suppose that there were N planets instead of three in the system from item A. N is an integer greater than 3. If the orbits were still circular, concentric, and coplanar, and the planets and star were all initially aligned, would they necessarily align again afterwards? Assume that the sidereal periods of all planets are rational numbers in terms of some unit period.

(c) (8 points) In the system from (a), if the three planets were not initially aligned with respect to the star, would they necessarily be perfectly aligned at some point? Again, use quantitative arguments to prove your answer.

(d) (4 points) Suppose that you are an astronomer who wants to use a telescope to observe the conjunction. Since you are a very skilled astronomer, you are going to build your own telescope. The only basic requirement you want to meet is that your telescope must be able to resolve the planets at the minimum separation during the conjunction. Calculate the value of all parameters of your telescope that are relevant for this goal. Do not try to calculate the values of any parameters that are not related to this requirement.

(e) (8 points) Calculate the total apparent magnitude of the planets together in the conjunction. Assume that the observers see Jupiter and Saturn as a single point in the sky, but Saturn is not covered (totally or partially) by Jupiter. For this item, neglect the atmospheric extinction, consider that the planets reflect isotropically, and consider that the albedos of both Jupiter and Saturn are equal to one. Also, in order to make the calculations simpler, assume that both Jupiter and Saturn were almost in opposition with respect to the Earth (even though this was not the case for this conjunction).

(f) (5 points) Calculate the difference in the magnitude of the conjunction at the zenith and at a zenith distance of 15^o. Assume that the zenith optical depth of Earth's atmosphere for visible light is 0.50.

- Mean orbital radius of Jupiter: 5.2 AU
- Mean orbital radius of Saturn: 9.5 AU
- Radius of Jupiter: 7.1492×10^7 meters
- Radius of Saturn: 5.8232×10^7 meters
- Apparent magnitude of the Sun: -26.74
- Central wavelength of visible light: 550 nm

Solution:

(a) It is possible to write the following expression for the synodic period between the first two planets ($S_{1,2}$, in which planet 1 is the closest to the star, and T_n represents the sidereal period of the n^{th} planet):

$$\frac{1}{S_{1,2}} = \frac{1}{T_1} - \frac{1}{T_2}$$
$$S_{1,2} = \frac{T_1 T_2}{T_2 - T_1}$$

Likewise, the synodic period between planets 1 and 3 ($S_{1,3}$) is the following:

$$S_{1,3} = \frac{T_1 T_3}{T_3 - T_1}.$$

It is important to notice that both synodic periods are rational numbers. All sidereal periods are rational numbers, and by definition, the subtraction or division of two rational numbers must result in a rational number, so both $S_{1,2}$ and $S_{1,3}$ are rational.

If we assume that the planets are aligned at $t = 0$, planets 1 and 2 must be aligned at all instants $t = m \cdot S_{1,2}$, in which $m \in \mathbb{N}$. Likewise, planets 1 and 3 must be aligned at all instants $t = n \cdot S_{1,3}$, in which $n \in \mathbb{N}$. Therefore, the following condition must be met for a triple alignment:

$$m \cdot S_{1,2} = n \cdot S_{1,3}$$
$$\frac{m}{n} = \frac{S_{1,3}}{S_{1,2}}$$
$$\frac{m}{n} = r.$$

Since r is the ratio between two rational numbers, it must also be a rational number. By definition, every rational number might be expressed as the ratio between two integers. In this case, since r must be positive, the two integers must have the same sign. Therefore, this equation has solutions in which $m, n \in \mathbb{N}$, so the planets will align periodically after $t = 0$.

(b) It is possible to expand the alignment condition from item A to more planets:

$$m_1 \cdot S_{1,2} = m_2 \cdot S_{2,3} = ... = m_n \cdot S_{n,n+1}.$$

In this equation, all m_n terms are natural numbers greater than zero.

Again, it is possible to use the definition of a rational number to solve this problem. As demonstrated on item A, if the sidereal periods are rational numbers, the synodic periods must be rational as well. Since a rational number might be expressed as a ratio between two integers, the product $m_n \cdot S_{n,n+1}$ is an integer for certain values of m_n. Therefore, there are values of m_n for which all $m_n \cdot S_{n,n+1}$ terms in the alignment equality are natural numbers. By definition, any set of natural numbers has a least common multiple (LCM), and an infinite number of common multiples. Therefore, it is possible to multiply all $m_n \cdot S_{n,n+1}$ terms by an integer factor to obtain the LCM. If all terms are equal to the LCM, the alignment equality is true, which proves that all planets are aligned at an instant later than $t = 0$.

(c) In order to obtain an expression for the instants $t_{1,2}$ in which the first two planets are aligned, it is possible to write the following formula, in which m $\in \mathbb{N}$ and $\theta_{i,n}$ corresponds to the initial angular position of the n^{th} planet:

$$\theta_{i,1} + \frac{2\pi}{T_1} t_{1,2} + 2\pi m = \theta_{i,2} + \frac{2\pi}{T_2} t_{1,2}$$

$$t_{1,2} \left(\frac{1}{T_1} - \frac{1}{T_2} \right) = \left(\frac{\theta_{i,2} - \theta_{i,1}}{2\pi} - m \right)$$

$$t_{1,2} = \frac{1}{S_{1,2}} \left(\frac{\theta_{i,2} - \theta_{i,1}}{2\pi} - m \right)$$

Likewise, the formula for the alignment between planets 1 and 3 will be the following:

$$t_{1,3} = \frac{1}{S_{1,3}} \left(\frac{\theta_{i,3} - \theta_{i,1}}{2\pi} - n \right).$$

In this formula, n is a natural number.

Therefore, it is possible to write the following equality for the alignment between three planets:

$$\frac{1}{S_{1,2}} \left(\frac{\theta_{i,2} - \theta_{i,1}}{2\pi} - m \right) = \frac{1}{S_{1,3}} \left(\frac{\theta_{i,3} - \theta_{i,1}}{2\pi} - n \right)$$

In order to simplify the formula, it is possible to group a few variables together:

$$\alpha = \frac{S_{1,3}}{S_{1,2}}$$

$$\beta = \frac{\theta_{i,2} - \theta_{i,1}}{2\pi}$$

$$\gamma = \frac{\theta_{i,3} - \theta_{i,1}}{2\pi}$$

Since α, β, and γ are the result of consecutive subtractions and divisions of rational numbers, they are also rational numbers. However, they are not necessarily rational.

It is possible to rewrite the formula using α, β, and γ:

$$\alpha(\beta - m) = (\gamma - n)$$

$$\alpha m - n = \alpha\beta - \gamma$$

It is possible to define a new variable δ to simplify the expression even more:

$$\alpha m - n = \delta$$

From this expression, it is clear that there are values of α and δ for which there are no solutions in which both m and n are natural numbers.

The easiest way to demonstrate this is with a simple counterexample. Suppose that $\alpha = 2$ and $\delta = 3.5$.

$$2m - n = 3.5$$

If both 2n and m are integers, the result of the subtraction should necessarily be an integer. However, since 3.5 is not an integer, there are no solutions for which both m and n are integers. Therefore, if the three planets are not initially aligned, they might never have a triple conjunction.

It is important to highlight that α is a function only of the planets' periods, but δ is a function of both the periods and the initial angular positions. Therefore, the method of choosing arbitrary values for a counterexample is valid.

Note: For the first three items of this question, students were allowed to assume that the periods of the planets are rational numbers.

However, the period of a body in a circular motion is not necessarily a rational number. It must be a real number, but it might also be an irrational number. For instance, consider a circular movement with an angular velocity of 1 rad/s. In this case, the period would be equal to 2π, which is an irrational number.

The reason why this question assumes that the periods are rational numbers is that it is impossible to measure periods with an infinite number of significant figures. On the aforementioned example, although 2π is an irrational number, 6.28319 is rational. Since we can only measure periods with a limited number of significant figures, even our most precise measurement for the period of any planet will still be a rational number.

It is also important to notice that in a real life situation, it is essential to consider that planets are not point particles and take into account the radius of each planet. Consider a hypothetical system with three planets with periods of 0.5 year, 1 year, and 2.000000000000000000000001 years. It is clear that if the planets are initially aligned, they will be pretty much aligned again in two years (considering that the third period is equal to approximately 2 years). However, because the third period is not exactly 2 years, the time interval until the next perfect alignment is very long. In this case, it makes much more sense to consider that the period between the alignments is equal to two years, not to a very large number.

(d) In this the, the angular resolution must be less of equal to 6.11'. Besides the wavelength, the diameter is the only parameter that affects the angular resolution. It is possible to use the following formula to calculate the diameter:

$$\theta = 1.22\frac{\lambda}{D}$$

$$D = 1.22\frac{\lambda}{\theta}$$

Since $\theta \leqslant 6.11'$ and visible light is centered at 550 nm:

$$D \geqslant 1.22\frac{5.50 \times 10^{-7}}{6.11\pi/(60 \times 180)}$$

$$D \geqslant 3.78 \times 10^{-4} m$$

$\therefore$ The diameter of the telescope must be greater or equal to 3.78×10^{-4} m. In other words, basically any telescope you could possibly build will meet this requirement.

(e) Solar flux that arrives at Jupiter and Saturn:

$$F_J = \frac{L_\odot}{4\pi r_J^2}$$

$$F_S = \frac{L_\odot}{4\pi r_S^2}$$

Flux from Jupiter and Saturn that arrives at the Earth:

$$F_{\oplus 1} = \frac{F_J \pi R_J^2}{4\pi (r_J - r_\oplus)^2} + \frac{F_S \pi R_S^2}{4\pi (r_S - r_\oplus)^2}$$

$$F_{\oplus 1} = \frac{L_\odot R_J^2}{16\pi r_J^2 (r_J - r_\oplus)^2} + \frac{L_\odot R_S^2}{16\pi r_S^2 (r_S - r_\oplus)^2}$$

$$F_{\oplus 1} = \frac{L_\odot}{16\pi} \left(\frac{R_J^2}{r_J^2 (r_J - r_\oplus)^2} + \frac{R_S^2}{r_S^2 (r_S - r_\oplus)^2} \right)$$

Solar flux that arrives at the Earth:

$$F_{\oplus 2} = \frac{L_\odot}{4\pi r_\oplus^2}$$

Ratio between the fluxes:

$$\frac{F_{\oplus 1}}{F_{\oplus 2}} = \frac{r_\oplus^2}{4} \left(\frac{R_J^2}{r_J^2 (r_J - r_\oplus)^2} + \frac{R_S^2}{r_S^2 (r_S - r_\oplus)^2} \right)$$

$$\frac{F_{\oplus 1}}{F_{\oplus 2}} = \frac{(1.496 \times 10^{11})^2}{4} \left(\frac{(7.1492 \times 10^7)^2}{5.2^2 \times 4.2^2 \times (1.496 \times 10^{11})^4} + \frac{(5.8232 \times 10^7)^2}{9.5^2 \times 8.5^2 \times (1.496 \times 10^{11})^4} \right)$$

$$\frac{F_{\oplus 1}}{F_{\oplus 2}} = 1.2551 \times 10^{-10}$$

Using Pogson's Law:

$$\begin{aligned} m_{\text{Conjunction}} - m_\odot &= -2.5 \times \log \left(\frac{F_{\oplus 1}}{F_{\oplus 2}} \right) \\ m_{\text{Conjunction}} &= -2.5 \times \log(1.2551 \times 10^{-10}) - 26.74 \\ m_{\text{Conjunction}} &= -1.99 \end{aligned}$$

∴ The apparent magnitude of the great conjunction is equal to -1.99.

Note: While students were supposed to assume that an observer would see Jupiter and Saturn as a single point in the sky, the minimum separation of 6.11' corresponds to about 1/5 of the Moon's angular diameter, so the human eye can easily resolve this angular separation.

(f) The optical depth (τ) is defined by the following expression, in which κ is the opacity coefficient, ρ is the density of the medium, and s is the distance travelled in the medium:

$$\tau = \kappa \rho s$$

Using the flat atmosphere approximation, which works well for small zenith distances, it is possible to calculate the optical depth of the Earth's atmosphere at a zenith distance of 15^o (τ_{15}). The only factor that varies in this case is the distance travelled, which is equal to $d_z \times sec(15^o)$, in which d_z is the distance travelled for a zenith distance of 0^o.

Therefore, is is possible to use the zenith optical depth τ_z to calculate τ_{15}:

$$\tau_{15} = \tau_z \times sec(15^o)$$
$$\tau_{15} = \tau_z \times sec(15^o)$$
$$\tau_{15} = 0.5 \times sec(15^o)$$
$$\tau_{15} = 0.5176$$

Considering that the ratio between the flux outside the atmosphere and the flux after atmospheric extinction is by definition the exponential function of the optical depth, it is possible to calculate the ratio between the fluxes for the zenith and for a zenith distance of 15^o:

$$\begin{aligned} \frac{F_{15}}{F_z} &= \frac{F_0 \cdot e^{-\tau_{15}}}{F_0 \cdot e^{-\tau_z}} \\ &= e^{\tau_z - \tau_{15}} \\ &= e^{0.5-0.5176} \\ &= 0.9825. \end{aligned}$$

Now, it is possible to use Pogson's Law to calculate the difference in magnitude:

$$\begin{aligned} \Delta m &= -2.5 \cdot \log\left(\frac{F_{15}}{F_z}\right) \\ &= -2.5 \cdot \log(0.9825) \\ &= 1.92 \times 10^{-2}. \end{aligned}$$

$\therefore$ The difference in magnitude between the Great Conjunction seen at a zenith distance of 15° and the Great Conjunction seen at the zenith corresponds to 1.92×10^{-2}.