2021 National Astronomy Competition

1 Instructions (Please Read Carefully)

The top 5 eligible scorers on the NAC will be invited to represent USA at the next IOAA. In order to qualify for the national team, you must be a high school student with US citizenship or permanent residency.

This exam consists of 3 parts: Short Questions, Medium Questions and Long Questions.

The maximum number of points is 180.

The test must be completed within 2.5 hours (150 minutes).

Please solve each problem on a blank piece of paper and mark the number of the problem at the top of the page. The contestant's full name in capital letters should appear at the top of each solution page. If the contestant uses scratch papers, those should be labeled with the contestant's name as well and marked as "scratch paper" at the top of the page. Scratch paper will not be graded. Partial credit will be available given that correct and legible work was displayed in the solution.

This is a written exam. Contestants can only use a scientific or graphing calculator for this exam. A table of physical constants will be provided. **Discussing the problems with other people is strictly prohibited in any way until the end of the examination period on March 20th.** Receiving any external help during the exam is strictly prohibited. This means that the only allowed items are: a calculator, the provided table of constants, a pencil (or pen), an eraser, blank sheets of papers, and the exam. No books or notes are allowed during the exam. Exam is proctored and recorded. You are expected to have your video on at all times.

After reading the instructions, please make sure to sign, affirming that:

1. All work on this exam is mine.
2. I did not receive any external aids besides the materials provided.
3. I am not allowed to discuss the test with others throughout the period of this examination.
4. Failure to follow these rules will lead to disqualification from the exam.

Signed: ______________________________

2 Short Questions

1. (**5 points**) Jupiter emits more energy to space than it receives from the Sun. The internal heat flux of Jupiter can be quantified by the "intrinsic" temperature of the planet T_{int}. The effective temperature T_{eff} of a planet is related to its intrinsic temperature and equilibrium temperature T_{eq} by $T_{\text{eff}}^4 = T_{\text{eq}}^4 + T_{\text{int}}^4$. Given that Jupiter's albedo is 0.5, its emissivity is 1, its average separation from the Sun is 5.2 AU, and its effective temperature is 134 K, estimate its intrinsic temperature in Kelvin. You may use the Sun's surface temperature equal to 5777 K.

2. (**5 points**) The convection zone of the sun is the major region of the solar interior that is closest to the surface. It is characterized by convection currents that quickly carry heat to the surface. As a pocket of gas rises, it expands and becomes less and less dense. For it to continue to rise, the temperature gradient in the sun must be steeper than the adiabatic gradient, which is the temperature that the gas would have if it were allowed to expand without any heat input.

   In the sun, the adiabatic gradient satisfies $T \propto p^{0.4}$, where T is the temperature and p is the pressure at any given point.

   The bottom of the convection zone is about 200,000 kilometers beneath the surface of the sun, and has a temperature of about 2×10^6 K and a density of about $200\,\text{kg/m}^3$. Estimate an upper bound for the temperature of the convection zone where the density is $1.2\,\text{kg/m}^3$ (the density of air). You may assume the ideal gas law holds in the convective zone.

3. (**5 points**) Galaxies are very hard to spot, even those that are nearest to us. For instance, Andromeda, despite having an apparent magnitude of 3.44, appears very "dim" in the sky. This is because its light is very spread out, since its solid angle in the sky is so large (around 3 times that of the Sun!).

   Hence, it is often useful to use the surface magnitude of a galaxy, defined as the magnitude that a certain solid angle of that galaxy has. It is usually measured in $mag/arcmin^2$.

   Show that, in a non expanding universe, the surface magnitude is independent of the distance to the galaxy.

4. (**5 points**) An Earth satellite has the following position ($\vec{r}$) and velocity ($\vec{v}$) vectors at a given instant:

$$\vec{r} = 7000\hat{\imath} + 9000\hat{\jmath}(\text{km})$$

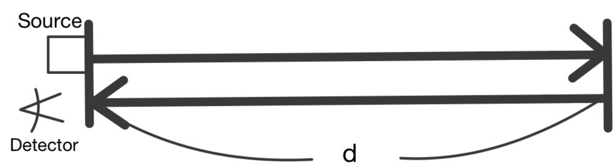
$$\vec{v} = -2\hat{\imath} + 5\hat{\jmath}\ (\text{km/s})$$

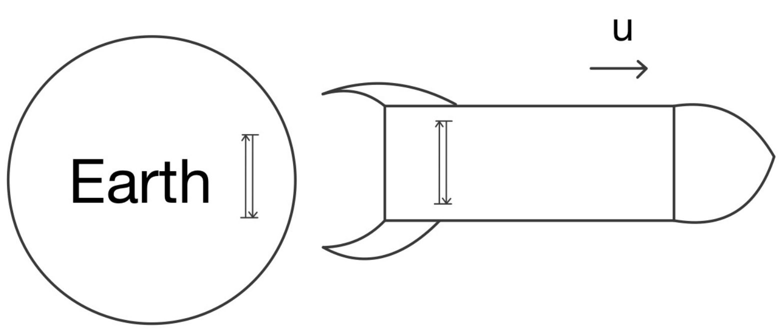
   Calculate the eccentricity of the satellite orbit. Hint: The eccentricity of the orbit is related to total energy E and angular momentum L as $e = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}$; where M is Earth's mass and m is the mass of the satellite.

5. (**5 points**) An astronomer who lives in Chicago ($\phi = 41.88^o N$; $\lambda = 87.63^o W$) was very bored during the day of the winter solstice in the Northern hemisphere, so he started thinking about the sunset. The astronomer could not wait to see the sunset on that day. Considering that the true solar time at his location was 2:30 pm, how long did he have to wait to see the sunset? The declination of the sun on winter solstice is $\delta = -23.44^o$.

3 Medium Questions

1. (**20 points**) To measure the time accurately outside the Earth, the engineers build a special clock, with the design as follows: there is a source of light that sends light particles (photons) straight to the reflector that is located at the distance d away from the source. The reflector sends the photons back to their starting point, where there is a detector. This can measure the time accurately, because the speed of light c is constant everywhere. Then a group of engineers built a spaceship with this special clock inside. This spaceship with a clock started to move really fast at the speed u. While the observer in the spaceship reported no issues with the clock inside the spaceship, the observer on the Earth has noticed that the clock is functioning differently in a fast moving spaceship than it is on Earth.





   (a) Given that the clock is at rest, what is the total traveling time (Δt_E) of a photon from its source back to the detector?

   (b) What is the total distance traveled by a photon d_γ from the source back to the detector on the spaceship moving at the speed u away from the Earth? (Here, we denote that the total traveling time of the photon as Δt_S)

   (c) What is the total time Δt_S of a photon as it travels from the source to the detector on the moving spaceship? Answer in terms of d, c, and $\beta = \frac{v}{c}$.

   (d) If we relate Δt_E (non-moving frame) to Δt_S (moving frame), as follows: $\Delta t_S = \gamma \Delta t_E$, what does γ equal to? What is significant about the range of γ?

   (e) So far, we have only analyzed the motion on the perspective of an observer on the Earth. From the perspective of an observer on the moving spaceship, how do the time on the spaceship $\Delta t_S{}'$ and the time on the Earth $\Delta t_E{}'$ relate to each other?

   (f) What can we conclude about the relative passing on time on two different frames that are relatively in motion to one another?

2. (**30 points**) You want to send a rocket with an instrument to analyze the atmosphere of Jupiter. In order to get there, you decide to use a Hohmann transfer orbit. $r_E = 1$ AU and $r_J = 5$ AU represent the radii of Earth's and Jupiter's circular orbits around the Sun, respectively. m, M_E, M_J, and M_S represent the masses of your rocket, Earth, Jupiter, and Sun, respectively. Ignore planetary gravitational influences. You may use any other variables you would like if you clearly define them first. Refer to the figures at the end of the question. Show your work for all derivations.

(a) Explain which two (relevant) physical quantities are conserved during this transfer orbit. Write down their statements mathematically.

(b) How long will it take to reach Jupiter?

(c) Halfway through its path to Jupiter, an unrealistic comet passes right next to your rocket and its icy tail freezes your rocket fuel. What is the **maximum** amount of time that you can afford to pass until you need the fuel to be once again unfrozen?

(d) Knowing that this comet will come in the way, your colleague suggests a bi-elliptic transfer orbit instead, with a peak distance of $12r_E$. Write equations describing how long it will now take to reach Jupiter. Will this solution always avoid the comet?

Now that you've compared the orbital times, you want to try and calculate the difference in efficiency.

(e) Derive the δv for each orbital transition in the Hohmann transfer, and sum them to find the total δv.

(f) Derive the δv for each orbital transition in the Bi-elliptic transfer, and sum them to find the total δv.

(g) Factoring in all your previous results, which transfer would you like to use? Why?

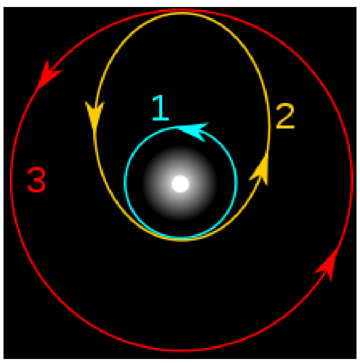


Figure 1: Hohmann Transfer

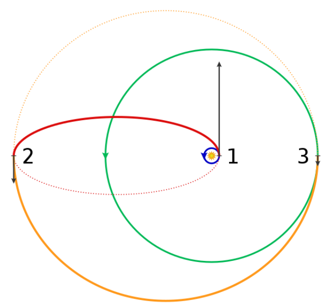


Figure 2: Bi-elliptic Transfer

3. (**30 points**) A space station of mass m is orbiting a planet of mass M_0 on a circular orbit of radius r. At a certain moment, a satellite of mass m is launched from the space station with a relative velocity $\overrightarrow{w}$ oriented towards the center of the planet. Assume that $w < \sqrt{\frac{GM_0}{r}}$.

(a) Justify the shape of the satellite's orbit after launching and, for the satellite-planet system, determine the following quantities:

(1) Satellite's velocity relative to the planet, immediately after launch, v

(2) Total angular momentum of the satellite-planet system, $L_{P,Sat}$

(3) Satellite's orbit semi-major and semi-minor axes, a_{Sat} and b_{Sat}

(4) Satellite's orbit eccentricity, ϵ_{Sat}

(5) Apogee and perigee distances, $r_{max,Sat}$ and $r_{min,Sat}$

(6) Satellite's minimum velocity, $v_{min,Sat}$ and maximum velocity $v_{max,Sat}$ on it's orbit

(7) Total energy of the satellite-planet system, $E_{Sat,P}$.

(b) Determine the shape of the space station's orbit relative to the planet, after the satellite was launched.

4 Long Questions

1. (**40 points**) In the very early universe, everything is in thermodynamic equilibrium and particles are freely created, destroyed, and converted between each other due to the high temperature. In one such process, the reaction converting between neutrons and protons happens at a very high rate. In thermal equilibrium, the relative number density of particle species is given approximately by the Boltzmann factor:

$$n_i \propto \exp\left[-\frac{E_i}{k_B T}\right],$$

where $E_i = m_i c^2$ is the rest energy. Additionally, the temperature during the radiation-dominated early universe is given by $T(t) \approx 10^{10}\,\mathrm{K}\left(\frac{t}{1\,\mathrm{s}}\right)^{-1/2}$, where t is the time since the Big Bang.

(a) (4 points) At a temperature where $k_B T \approx 0.8\,\mathrm{MeV}$, known as the *freeze-out* temperature, the neutrino interactions essentially stop, preventing further conversion between protons and neutrons.

i. (2 points) About how long after the Big Bang did this occur?

ii. (2 points) At the freeze-out temperature, what was the equilibrium ratio of the number density of neutrons to that of protons?

(b) (3 points) Free neutrons are unstable, and decay into protons with a characteristic decay time of $\tau = 886\,\mathrm{s}$ (the time for which the number of neutrons drops to $1/e$ of the original amount). Given that helium nuclei only formed $t_{nuc} = 200\,\mathrm{s}$ after freezing out, what was the ratio of the number density of neutrons to that of protons when the helium nuclei formed?

(c) (7 points) While trace amounts of several small nuclei were formed during Big Bang Nucleosynthesis (BBN), assume that all neutrons go into forming helium-4.

i. (5 points) After the helium nuclei formed, what was the ratio of the number of helium-4 nuclei to the number of hydrogen nuclei?

ii. (2 points) Approximating the mass of helium-4 as 4 times that of H (for this part only), what fraction of baryonic mass in the universe is helium?

If you weren't able to solve part (c), assume reasonable values for the initial mass fractions of hydrogen and helium for future parts.

(d) (2 points) Albert the Astronomer claims that in older galaxies, the mass fraction of hydrogen should gradually be increasing, as neutrons slowly continue to decay into protons. Is his claim correct? If not, explain.

(e) (7 points) Suppose a certain region of a galaxy has a density of $10^{-19}\,\mathrm{kg/m^3}$ and is composed of 70% hydrogen and 30% helium-4 by mass (ignore any heavier elements). Because the region is gravitationally bound, this density doesn't change significantly with the expansion of the universe; approximate it as constant. Assume hydrogen is converted into helium by the fusion reaction:

$$4\,{}^{1}\mathrm{H}^{+} + 2\mathrm{e}^{-} \rightarrow {}^{4}\mathrm{He}^{2+} + 2\nu_e,$$

where the electron e^- and electron neutrino ν_e are of negligible mass. ^{4}He has a mass of $m_{He} = 3728.4\,\mathrm{MeV/c^2}$

i. (4 points) Over the entire time since BBN, how much energy does this process release per cubic kiloparsec? Give your answer in joules per cubic kiloparsec.

ii. (3 points) Assuming the age of the universe is 13.8 billion years, calculate the average luminosity density in solar luminosities per cubic kiloparsec.

Let's go back and explore how we arrived at the number $t_{nuc} \approx 200\,\mathrm{s}$, the time at which Big Bang nucleosynthesis began. Let's define t_{nuc} as the time at which half the neutrons fused with protons into deuterium (^{2}H), as deuterium fusion is the first step in BBN. From the Maxwell-Boltzmann equation, the relative abundances of deuterium, protons and neutrons is given by

$$\frac{n_D}{n_p n_n} = 6\left(\frac{m_n k_B T}{\pi \hbar^2}\right)^{-3/2} \exp\left(\frac{B_D}{k_B T}\right),$$

where $B_D = (m_p + m_n - m_D)\, c^2 = 2.22\,\text{MeV}$ is the energy released in a deuterium fusion reaction.

(f) (3 points) The number density of photons is given by $n_\gamma = 0.243\left(\frac{k_B T}{\hbar c}\right)^3$. Find an expression for the number density of protons n_p in terms of the temperature T and the baryon to photon ratio η. You may use your answer to part (b).

(g) (3 points) Find the present-day baryon to photon ratio. The CMB temperature is 2.725 K, and the present-day density parameter for baryonic matter is $\Omega_{b,0} = \frac{\rho_{b,0}}{\rho_{c,0}} = 0.04$. ρ_c is the critical density of the universe, which is the density required for a flat universe; it is given by $\rho_c = \frac{3H^2}{8\pi G}$. Use $H_0 = 70\,\text{km/s/Mpc}$.

(h) (8 points) Assuming the baryon to photon ratio is fixed since the Big Bang:

i. (5 points) Find an equation involving T_{nuc} (the temperature at time $t = t_{nuc}$) and known constants.

ii. (1 point) What temperature T_{nuc} does $t_{nuc} = 200\,\text{s}$ correspond to?

iii. (2 points) Verify that this temperature solves your equation in part (h)i.

(i) (3 points) The baryon to photon η is a remarkably small number. One possibility is that the universe happens to prefer photons significantly over baryons. Another possibility is that a great number of quark-antiquark pairs were created in the early universe via pair production ($\gamma + \gamma \rightleftharpoons q + \bar{q}$), and a slight asymmetry of quarks over antiquarks produced a large number of photons during quark-antiquark annihilation, leaving over a small number of quarks to form into protons and neutrons. Find the quark-antiquark asymmetry

$$\delta_q \equiv \frac{n_q - n_{\bar{q}}}{n_q + n_{\bar{q}}} \ll 1$$

that would yield the baryon to photon ratio found in part (g).

2. (**35 points**) In 2020, during the day of the winter solstice for the Northern hemisphere, Jupiter and Saturn were at their minimum angular separation (approximately 6.11') during the Great Conjunction.

(a) (6 points) Consider a system with three planets in circular, concentric, and coplanar orbits around a star. Suppose that the three planets and the star are initially aligned. Will they necessarily align again after this moment? Prove your answer with quantitative arguments. Assume that the sidereal periods of all planets are rational numbers in terms of some unit period.

(b) (4 points) Suppose that there were N planets instead of three in the system from item A. N is an integer greater than 3. If the orbits were still circular, concentric, and coplanar, and the planets and star were all initially aligned, would they necessarily align again afterwards? Assume that the sidereal periods of all planets are rational numbers in terms of some unit period.

(c) (8 points) In the system from (a), if the three planets were not initially aligned with respect to the star, would they necessarily be perfectly aligned at some point? Again, use quantitative arguments to prove your answer.

(d) (4 points) Suppose that you are an astronomer who wants to use a telescope to observe the conjunction. Since you are a very skilled astronomer, you are going to build your own telescope. The only basic requirement you want to meet is that your telescope must be able to resolve the planets at the minimum separation during the conjunction. Calculate the value of all parameters of your telescope that are relevant for this goal. Do not try to calculate the values of any parameters that are not related to this requirement.

(e) (8 points) Calculate the total apparent magnitude of the planets together in the conjunction. Assume that the observers see Jupiter and Saturn as a single point in the sky, but Saturn is not covered (totally or partially) by Jupiter. For this item, neglect the atmospheric extinction, consider that the planets reflect isotropically, and consider that the albedos of both Jupiter and Saturn are equal to one. Also, in order to make the calculations simpler, assume that both Jupiter and Saturn were almost in opposition with respect to the Earth (even though this was not the case for this conjunction).

(f) (5 points) Calculate the difference in the magnitude of the conjunction at the zenith and at a zenith distance of 15^o. Assume that the zenith optical depth of Earth's atmosphere for visible light is 0.50.

- Mean orbital radius of Jupiter: 5.2 AU
- Mean orbital radius of Saturn: 9.5 AU
- Radius of Jupiter: 7.1492×10^7 meters
- Radius of Saturn: 5.8232×10^7 meters
- Apparent magnitude of the Sun: -26.74
- Central wavelength of visible light: 550 nm